THE MATHEMATICAL GAZETTE

The Journal of the Mathematical Association

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OF ELEMENTARY MATHEMATICS

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'I hold every man a debtor to his profession, from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and an ornament thereunto.'
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A LITTLE KNOWLEDGE IS A DANGEROUS THING

BY A. W. SIDDONS

About 1901 I saw something of a member of the firm which was laying out at South Harrow what is now known as the Piccadilly Line.

He explained to me how they marked out the central line and how on curves they used sighting posts well away from the central line.

I asked to what degree of accuracy they worked. He replied—"To absolute accuracy." I received his answer in silence.

About 1902 or 1903 I came home from School about noon one day to find this man waiting for me. They were engaged in driving pilot-tunnels for the Bakerloo Line; they had started tunnels from Baker Street and Waterloo and were expecting to meet the next day and thought in each tunnel to hear men working on the other tunnel; but not a sound was to be heard.

They were afraid that, owing to the difficulties of surveying in a tunnel with no sighting posts outside it, they were not going to meet.

He explained to me that to get a base line at Waterloo for instance, they had sunk two shafts behind public houses and dropped plumb lines down them, and surveyed through the public houses out into the roads.

Some bright spark had suggested that, owing to the rotation of the earth, the plumb lines would be deflected from the vertical and that that had put out their survey.

The questions he put to me were (i) was this the case? and (ii) what correction should they make?

I replied to (i) that the vertical is the direction of the plumb line, so there is no deflection. To (ii) I said that there was no correction

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to be made. He said he must go off at once and telephone my answers, as all work was at a standstill till they knew my replies.

Actually the two tunnels met the next day. Their centre lines were only about one or two inches apart horizontally and the same distance for level. A remarkably accurate piece of surveying under very difficult conditions.

A. W. S.

CHILDREN'S REMARKS

And their use in helping us to improve our teaching!

By C. D. Langford

Some children's remarks are funny and some are sad; most people realise that, but what interests me most is their instructive value in showing us how to improve our teaching.

I remember once telling a class that I thought they ought to have learned one item in arithmetic before. Tom E., a bright little fellow, promptly replied—"Oh yes Sir, we did it before with Mr. H. last year, but as we didn't understand it, we forgot it." I asked him if he *really* meant what he had said! He looked at me puzzled, so I pointed out that he had said that if he didn't understand anything, then he forgot it. The whole class laughed and agreed that generally speaking this was true.

This was in an A Form—very much an A Form, if I may be allowed to use such an expression, and it was interesting, therefore, when, during the same year I heard a new girl in a higher C Form whisper to her neighbour—"What is the use of trying to understand Mathematics? Mathematics wasn't *meant* to be understood!" I had something to say about this idea and was later amused, interested and pleased to hear this same girl say (some months later)—"Oh, is that what two sides and the included angle means? Miss J. was always talking about them, I often wondered what it meant!!" So apparently even Vera B. *liked* to understand ... but did not regard it as reasonable to expect to do so at all times! (Miss J. turned out to be the last teacher to take her for Mathematics in her last school—it would not be fair to call her a Mathematics Teacher since I learned that Mathematics had been taught by a different teacher in each form (i.e. the form teacher)—in fact I found that each teacher had insisted on a different method for such things as "finding the position of the decimal point in multiplication or division.) When Vera realised that I regarded her poor Mathematics as her misfortune due to bad luck and not her fault, she cheered up and soon made remarkable progress though she never actually showed any sign of genius.

In my very first term of teaching I remember asking a boy what 3 multiplied by one thousand was (it came up in a little problem); he looked at me blankly, after trying every help and hint I could think of I finally said "What is 3 multiplied by one and three noughts?" and was somewhat surprised when he replied in *indignant* tones,—"Oh Sir, three and three noughts, why didn't you say that before, then I should have known what you meant"! Poor lad, it seemed that he had been taught symbol juggling rather than Mathematics! But it gave me an idea. How often had *I* said "two over three 'when I meant 'two-thirds'? So, from then on I insisted that all fractions should be called by 'what they were' and not just by 'how they are written'!

The improvement in all their Mathematics and in Vulgar Fractions in particular, was soon so astonishing that it was the class that commented on it first. (In this connection we should consider the pupils' point of view. From infancy he has seen on the top of all his corrected work returned to him such things as $\frac{2}{5} + \frac{3}{5} = \frac{5}{10}$ so that he is apt to say later '2 over 5 plus 3 over 5 equals 5 over 10'; but if one insists on 'two-fifths plus three-fifths' he is more likely to say 'five-fifths' for their sum. If he says 'tenths', ask him if 2 apples plus 3 apples equals 5 apples or 5 apples apples!)

These, of course, are only a few examples, but they do, I hope, serve to show that many of our pupils' remarks can give us a clue as to possible improvements in our method. (Though actually the author himself had two accidents as an adult on top of polio as a boy and is now quite 'hors de combat' and so unable to do any teaching at all now!) C. D. L.

16 *Henrietta St., Girvan, Ayrshire*

THE PROBLEM BUREAU

By G. A. Garreau

The late Mr. A. S. Gosset Tanner conducted the Bureau for the Solution of Problems from 1928 almost up to the day of his death in 1953 and many members of the Association will remember with gratitude the help he gave so freely, and with undiminished vigour as the years went on. Mr. Gosset Tanner kept files of solutions to the papers for scholarships in *Mathematics* at Cambridge colleges. These files have been kept up to date, and the solutions are available to any member of the Association who, after a genuine attempt to solve a problem, has failed or has a solution with which he is not satisfied. The enquirer who asks for "solutions to all the questions

in the *X* group for the last five years" may meet with a cold reception. Apart from the fact that he should not be teaching this standard of work if he can't solve any of the questions, it is obvious that the coefficient of restitution of large masses of solutions is small, because they take a long time to read. Other members will be asking for some of the solutions before they are returned. For this reason, not more than six solutions are generally sent at a time.

In recent years, Cambridge Scholarship papers have included some questions which are not problems, and which, therefore, the Bureau declines to "solve". For example, 280/49/10 for the Clare Group, 1957: "A plank rests across a cylindrical barrel on flat ground and initially has one end on the ground. A man walks up the plank. Discuss qualitatively the possible resulting motions on the assumption that no slipping occurs. (For example, consider different ratios of masses and dimensions and consider the man taking short or long steps and moving irregularly.)" There have been complaints that it is very difficult to provide model solutions for such questions. No doubt this is the very reason that they are set. It must be remembered that the purpose of the papers is to select future mathematicians, not to provide problems whose solutions can be neatly filed away. As it happens, the first question on the same paper has aroused great interest: "A tennis match is played between two teams, each player playing one or more members of the other team. Further, (i) any two members of the same team have exactly one opponent in common; (ii) no two members belonging to the same team play all the members of the other team between them. Prove that two players who do not play each other have the same number of opponents. Deduce that any two players, whether belonging to the same team or different teams, have the same number of opponents."

Another question, 273/44/5 from the same group in 1956, reads as follows: "A region R of the plane is defined to be *convex* if for each pair of points A, B, both lying in R, the whole line segment between A and B also lies in R; R is defined to be a *polygonal convex region* if it is convex and bounded by a polygon.

An island forms a *polygonal convex region* of a flat earth. In times of emergency civilians are not allowed to be within d miles of the coast, the value of d varying with the emergency. Prove that the region to which at any time civilians are restricted is a polygonal convex region.

The capital is to be sited so as to lie in all possible civilian regions. Prove that, if R is bounded by a quadrilateral Q, this rule determines uniquely the site of the capital, provided Q is not a trapezium."

At the special request of an enquirer, a solution to this problem was prepared by a member of the Problem Bureau team of solvers, who was Senior Mathematical Master at a very great public school.

The enquirer complained that it was not his idea of a proof. Presumably he had expected something on the lines of Euclid.

In many cases, enquirers have returned solutions with suggestions for improvement. These suggestions are always welcome, and help us towards the unattainable goal of perfection. The Bureau can handle plenty more problems, so please do not hesitate to write (enclosing a stamped addressed envelope) to Dr. G. A. Garreau, 90, Wyatt Park Road, London S.W.2. if the Bureau can be of help.

G. A. GARREAU

HOW TO FIND MOMENTS OF INERTIA WITHOUT ACTUALLY INTEGRATING

BY D. E. RUTHERFORD

The evaluation of moments of inertia of simple solids is frequently regarded merely as a convenient source of problems in the integral calculus. From the point of view of the student of dynamics it is, however, perhaps even more instructive to make these calculations without any actual integration, for in this way he may obtain a deeper insight into the physical significance of the underlying concept. Whether this be so or not it seemed to the writer an amusing task (performed during a moment of inertia caused by a bacteriological infection) to see how many of the well known formulae could be derived by such methods.

In the sequel we find it convenient to lay the emphasis on the radius of gyration k rather than on the moment of inertia and we denote the moment of inertia about an axis PQ of a body of mass M by Mk_{PQ}^2. For instance,

$$Mk_{OZ}^2 = \int (x^2 + y^2)\, dm.$$

where dm is the element of mass all of which has coordinates (x, y). It is at once obvious that this element of mass may be redistributed in any way we please parallel to the z-axis without altering the value of k_{OZ}. Again, if M is composed of a number of masses $m_1, \ldots, m_n$ whose radii of gyration about the same line PQ are $k_1, \ldots k_n$ then $k_{PQ}^2 = \dfrac{\Sigma\, m_i k_i^2}{\Sigma\, m_i}$. It follows that if all the k_i are equal, even when n is infinite, each has the value k_{PQ}. From the fact that k has the dimension of a length it follows that if the dimensions of a body are multiplied by some factor, then k is multiplied by the same factor. In view of these remarks we can now formulate the five basic results upon which we base most of our arguments.

I For a lamina lying in the plane OXY

$$k_{OZ}^2 = k_{OX}^2 + k_{OY}^2.$$

II If G is the mass centre and if h is the distance between the parallel axes QA and GB then

$$k_{QA}^2 = k_{GB}^2 + h^2.$$

III k_{PQ}^2 is unaltered by displacing or by redistributing the mass of any element of the body, keeping the distance of that element from PQ unaltered.

IV If a body can be divided up into any number of parts so that each part has the same radius of gyration about PQ then this is also the radius of gyration of the whole body about PQ.

V If two similar bodies have similarly situated axes, the ratio of their radii of gyration about their respective axes is that of their linear dimensions.

Of the above, I and II are familiar enough and their justification is to be found in any text book. While III, IV, V may be well known to teachers and lecturers they are seldom familiar to pupils and students. It is hoped that the following lines will illustrate their usefulness.

Consider (Fig. 1) a uniform thin rod AB of length $2a$, with mid-

Fig. 1

point O, lying along the x-axis and let k be its radius of gyration about OY; then (by IV) k is also the radius of gyration of AO about OY. If G is the mid-point of AO, then (by II) the radius κ of gyration of AO about GH is given by $\kappa^2 = k^2 - (\frac{1}{2}a)^2$. But $\kappa^2 = \frac{1}{4}k^2$ (by V). Hence $k^2 = \frac{1}{3}a^2$.

Regarding a flat plate $PQRS$ (Fig. 2) on sides $2a$, $2b$ as made up of a large number of thin rods of length $2a$, we have (by IV) $k_{OY}^2 = \frac{1}{3}a^2$, and similarly $k_{OX}^2 = \frac{1}{3}b^2$. Thus (by I), we obtain $k_{OZ}^2 = \frac{1}{3}(a^2 + b^2)$. This last formula must also hold (by IV or by III) for a rectangular block which has $PQRS$ as one face.

Returning to the case of the plate $PQRS$ but in the case where $PQRS$ is square of side $2a$, $k_{OZ}^2 = \frac{2}{3}a^2$. From I we deduce that

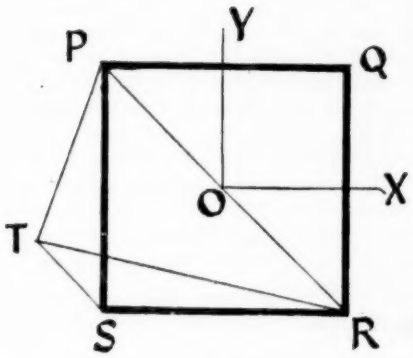

Fig. 2

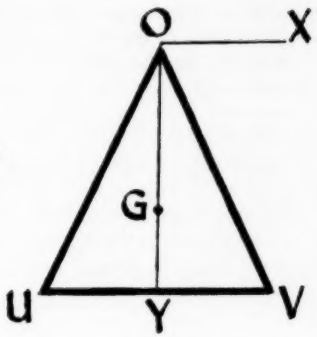

Fig. 3

$k_{OP}^2 = k_{OQ}^2 = \frac{1}{2}k_{OZ}^2 = \frac{1}{3}a^2$. It now follows (by IV) that $k_{OP}^2 = \frac{1}{3}a^2$ for the triangular plate PSR and also (by III) for any other triangular plate PTR with the same base PR and the same height $h = a\sqrt{2}$. For such a plate PTR we therefore have $k_{PR}^2 = \frac{1}{3}h^2$ and a further consideration of III shows that this formula holds good for any triangle of height h no matter what its base length may be. Thus for any uniform triangular plate OUV (Fig. 3) on base UV of height h we have $k_{UV}^2 = \frac{1}{6}h^2$, while (by II)

$$k_{OX}^2 = \tfrac{1}{6}h^2 - (\tfrac{1}{3}h)^2 + (\tfrac{2}{3}h)^2 = \tfrac{1}{2}h^2.$$

If OUV is isosceles of base $2c$, then $k_{OY}^2 = \frac{1}{6}c^2$ (by IV), since OUV is made up of two equal triangles on the same base OY.

For the whole isosceles triangle OUV we now have (by I)

$$k_{OZ}^2 = \tfrac{1}{2}h^2 + \tfrac{1}{6}c^2$$

and this formula will hold (by IV) for any regular polygon in the plane OXY with centre O composed of triangles similar to OUV. In fact, for a regular n-gon of side $2c$ we have only to substitute $h = c \tan\left(\frac{1}{2} - \frac{1}{n}\right)\pi$ in the above formula.

Consider the momental ellipse for such a plane polygon. Let OX, OY be principal axes at its centre. Then, if OL is the line $y = x \tan\theta$, a well known formula states that

$$k_{OL}^2 = k_{OX}^2 \cos^2\theta + k_{OY}^2 \sin^2\theta,$$

or

$$(k_{OL}^2 - k_{OY}^2) = (k_{OX}^2 - k_{OY}^2)\cos^2\theta.$$

If it can be shown that there are three distinct axes OL_1, OL_2, OL_3 in the plane OXY such that $k_{OL_1}^2 = k_{OL_2}^2 = k_{OL_3}^2$ then a simple calculation will show that $k_{OX}^2 - k_{OY}^2 = 0$. Then the momental ellipse is a circle and k_{OL}^2 has the same value for any line OL in the plane OXY. Furthermore (by I) $k_{OL}^2 = \frac{1}{2}k_{OZ}^2$, while the value of k_{OZ}^2 is quoted above. Indeed, every regular n-gon except the square clearly has at least three such axes OL_1, OL_2, OL_3 on account of the n-fold symmetry, while we have already seen that the square too has this property for a slightly different reason. We have therefore determined k_{OL}^2 for any line OL lying in the plane of a regular n-gon and passing through its centre.

Returning to the isosceles triangle OUV, we see that in the limiting case when the base of length $2c$ tends to zero

$$k_{OZ}^2 = \tfrac{1}{2}h^2.$$

Regarding a circular disc of radius a as composed of an infinite number of such triangles each of height a we obtain (by IV)

$$k_{OZ}^2 = \tfrac{1}{2}a^2$$

where OZ is the axis of the disc. It is immediate (by I) that

$$k_{OX}^2 = k_{OY}^2 = \tfrac{1}{4}a^2$$

where OX is a radius of the circle.

The well known fact that corresponding ordinates for the circle $x^2 + y^2 = a^2$ and for the ellipse $x^2/a^2 + y^2/b^2 = 1$ are in the ratio $a : b$ enables us to state (by III) that for this ellipse too $k_{OY}^2 = \frac{1}{4}a^2$. By symmetry we obtain $k_{OX}^2 = \frac{1}{4}b^2$ and consequently (by I) $k_{OZ}^2 = \frac{1}{4}(a^2 + b^2)$.

In the case of a hollow sphere of radius a and centre O, we have

$$M(k_{OX}^2 + k_{OY}^2 + k_{OZ}^2) = \int 2(x^2 + y^2 + z^2)\,dm$$

$$= \int 2a^2\,dm = 2a^2M,$$

from which we obtain

$$k_{OX}^2 = k_{OY}^2 = k_{OZ}^2 = \tfrac{2}{3}a^2.$$

In the case of a uniform solid sphere only the most determined will be able to refrain from integrating the last result. Avoiding integration, the calculations are lengthy and we content ourselves with sketching the method. Consider the pyramid with base $ABCD$ and vertex the origin, where $A = (a, 0, h)$, $B = (a, a, h)$, $C = (0, a, h)$, $D = (0, 0, h)$, and write $k_{OX}^2 = k_{OY}^2 = \alpha^2$, $k_{OZ}^2 = \beta^2$. A pyramid which is similar to $OABCD$ but of twice its linear size can be composed from two pyramids each equal to $OABCD$, a rectangular

block of height h and base area a^2, and two triangular prisms each of which is one half of the block just mentioned. Since the prisms can be related (by IV) to the rectangular block we can calculate the radii of gyration for the large pyramid either from those of its component parts or with the help of V. Equating these answers we find that $\alpha^2 = \frac{1}{5}(a^2 + 3h^2)$, $\beta^2 = \frac{3}{5}a^2$. Then

$$(k_{OX}{}^2 + k_{OY}{}^2 + k_{OZ}{}^2) = \tfrac{1}{5}(4a^2 + 6h^2)$$

so that for a pyramid with vertex O of height h and infinitesimal base

$$\int 2(x^2 + y^2 + z^2)\, dm = M(k_{OX}{}^2 + k_{OY}{}^2 + k_{OZ}{}^2) = \tfrac{6}{5}Mh^2.$$

Regarding a sphere as made up of an infinite number of such pyramids a slight adaptation of IV shows that these last formulae apply to sphere of radius h. Thus for the sphere

$$k_{OX}{}^2 = k_{OY}{}^2 = k_{OZ}{}^2 = \tfrac{2}{5}h^2.$$

D. E. R.

MATHEMATICA PRIMA

By H. V. Lowry

I believe that a useful contribution to the Gazette would be an occasional description of the excellent work done in Mathematics by the boys and girls of various schools. To start the ball rolling I am writing this article to tell of the very good work done at St. Dunstan's College, Catford, some of which has been published every year since 1952 in a special mathematical magazine of their own called "Mathematica Prima". This very interesting publication contains articles on a wide variety of topics nearly always including one or two on original work. I think I can indicate best what kind of work has been done at St. Dunstan's by giving the table of contents of the 1959 issue with a few explanatory notes of my own in brackets and following this by short extracts from a few items taken from different numbers of the magazine which have been of particular interest to me.

The contents of *Mathematica Prima* No: 10, 1959 were as follows:—

The Differential Analyser, by G. G. Allwood, R. J. Harvey and G. Hipperson. {A description of a small analyser made from meccano parts for the School Speech Day in 1958.}

Book review, by R. G. Miller. {"How to Solve it" by G. Polya.}

The Story of Leonard Euler, by A. N. Morey.

A new look at radical axes, by J. M. Hill.

How many squares are there in a chessboard of k, m units? by D. M. Jackson. {It is proved that the number of squares is

$\frac{1}{6}k\,(k+1)(3m-k+1)$ if $k \leqslant m$; and this is followed by a proof that for a cuboid of units k, m, n, $k \leqslant m \leqslant n$, the number of cubes is $\frac{k}{12}[12kmn - (k-1)\{(k+1)(2m+2n-k)+6mn\}]$.}

A theorem on the triangle, by J. B. Sampson. {This theorem is one of the items given in more detail later.}

A formula for $\sum_1^n x^k$, by D. M. J. {It is proved that

$$\left(k+1!\right)\sum_{x=1}^{n} x^k = k!\,[(n+1)^{k+1}-1] - \sum_{m=1}^{k}(k-m)!\,u_m[(n+1)^{k+1-m}-1],$$

where u_m $(m = 1, 2 \ldots k)$ are successively defined by

$$k!\binom{k+1}{k-r} = \sum_{\lambda=1}^{r}(k-\lambda)!\binom{k+1-\lambda}{k-r}u_\lambda.\}$$

Universal gravitation applied to the major planets, by G. G. Allwood. {A proof that what goes up must come down from a relativity standpoint.}

Problem for this issue, set by B. S. Maitland. {This problem is "Translate into English a passage of which the first line is $\ni t_0$ $\ni t > t_0$, $\ni \delta g$, $\delta g \equiv G.\ @\ t_0 = 1$ F.D.,"}

Some items in these ten numbers of the magazine that have particularly interested me are as follows:—

(1) The theorem mentioned above by J. B. Sampson. This theorem is:—

If A', B', C' are the mid-points of the sides of a triangle ABC and X, Y, Z are the points of contact of the in-circle with the sides, then XY, ZY meet $A'C'$ on the internal bisectors of the angles at B and C respectively and similarly for the other four points in which YZ, XZ meet $A'B'$ and XZ, XY meet $B'C'$. The proof given is trigonometrical mainly using the formulae for the tangents of the half angles. This theorem leads on to another, namely:— If in a triangle ABC with circumcentre O, A', B', C' are the mid-points of BC, CA, AB and $A'O$, $B'O$ meet AC, BC in N_1, N_2 then N_1N_2 is perpendicular to CO and hence parallel to the tangent at C to the circle ABC. Further if L_1, L_2, M_1, M_2 are similarly defined then L_1L_2, M_1M_2, N_1N_2 form a triangle whose vertices are mid-points of the sides of the triangle ABC.

I understand these theorems were discovered by drawing an enormous triangle and every associated line that could be thought of and looking for unexpected coincidences.

(2) Construction for a triangle equal in area to a given quadrilateral (in No: 3):—$ABCD$ is the quadrilateral and its diagonals

meet in O. Produce AC to E so that $CE = AO$. Then the triangle DBE is equal in area to the given quadrilateral. (This construction was "invented" by its author R. Haythornwaite because the official one had been forgotten.)

(3) Divisibility test for 17 (in No: 7):—Double the number formed by removing the last two digits and subtract from the last two digits. Repeat this process until the number is small enough to divide by 17. Then the remainder is the same as the remainder obtained when the original number is divided by 17 (negative numbers occur in this process). This test came to the author P. J. Roberts when acting as a rather bored steward at a lower school meeting. It has already appeared as a Note in the *Gazette*.

(4) A wallpaper curve (in No: 7):—This class of curve is obtained by substituting trigonometrical functions in equations of well-known curves. The example given is the curve

$$\tan^3 x + \tan^3 y = 3 \tan x \tan y.$$

(5) A probability problem (in No: 5):—A man who tells the truth two times in three draws a card from a full pack of 52 cards and says it is the five of diamonds. A second man who tells the truth one time in two sees the card independently and says it is a diamond. Show that the probability of the card actually being the 5 of diamonds is 107/127. (Cambridge Scholarship Question.)

Finally I must given another problem which I feel deserves an Oscar for problems of this type.

(6) A fishy problem (in No: 8):—A party of 7 fathers, each with his son, went on a fishing trip. A contest was arranged between them. For the convenience of the judges the fathers were numbered 1 to 7, the sons were lettered from A to G. The average weight in ounces of the fish caught by each person was the same as the number of fish he caught. The fathers eventually won with 99 more fish than the sons. Each father's catch weighed 85 lb. 5 oz. more than his son's.

Mr. Perch caught the most fish.

Mr. Shark caught 1 less than B.

Mr. Carp caught 42 lb. less fish than No: 3.

Edward caught the least fish.

Bert caught more than 40.

C caught 11 times as many fish as D.

Charlie had 101 less than Mr. Rudd.

No: 5 caught 3 more than his son, F.

Alf caught 40 more than Mr. Tench's son.

Mr. Gudgeon had 3 less fish than E.

Twice Bert's catch added to that of Edward was the same as No: 4's son.

Mr. Tench and Charlie together caught as many fish as No: 1.

Donald caught 204 less than Fred.
No: 4's son was *A*.
C's father was No: 7.
Mr. Gudgeon caught 10 less than No: 2.

The other members of the party were George and Mr. Pike. Make a list of the fathers and sons showing the correct relationships and the correct numbering and lettering used in the competition.

I have not given any of the solutions for the above theorems and problems because I am sure members of the Association will enjoy finding the solutions for themselves or asking their pupils to find them.

I am much indebted to Mr. G. Matthews, Senior Mathematics Master at St. Dunstan's College for information he has given me about the articles in Mathematica Prima. I feel sure a great deal of the inspiration for the work done by the boys at the College must come from him.

H. V. L.

THE PLAITED DODECAHEDRON

By James Brunton

Mr. Pargeter deserves our thanks for introducing us to the fascinating art of plaiting polyhedra (May 1959) and I would like to indulge in the luxury of two comments on what he said.

(i) In para. 5.7 Mr. Pargeter says, "Chartwell Paper is excellent for aiding the construction of nets using only 60° or 120° etc." but I use a simpler method, which has the advantages of being much cheaper and of being something that the pupil can prepare himself. The paper folding in the following diagrams is very accurate if carefully done and provides some good revision for the abler pupils. Solid lines indicate the edges of the paper and thin lines the folds. Folds 2 & 3 are then repeated from the right-hand side to complete this basic triangle and it can then be sub-divided into an all-over pattern of smaller ones, either by folding *A* & *B* on to the centroid *G* and then parallel folds, or into even smaller triangles by folding the height into half and then quarters. The completed pattern is used by my pupils as the template for nets in the way Mr. Pargeter suggests, by pricking through on to plain paper, or else, as we sometimes do, by direct cutting.

(ii) In his note on model 19, after indicating how the compound plait for the dodecahedron can be developed, Mr. Pargeter says "No substantially simpler plait for the dodecahedron appears to be

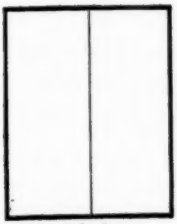

Fig. 1

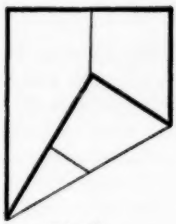

Fig. 2

Fig. 3

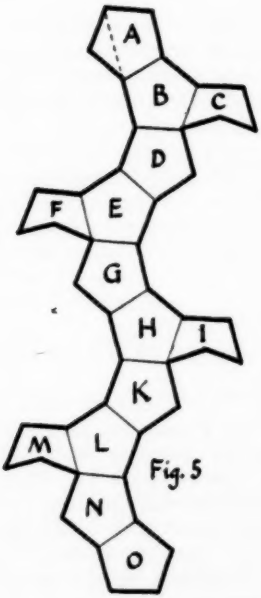

Fig. 5

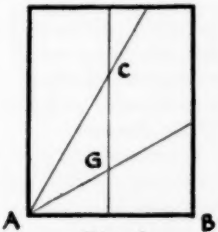

Fig. 4

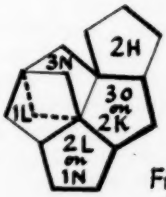

Fig. 6

FIG. 1. The first fold is the perpendicular bisector of the base-line.

FIG. 2. This fold is best done by keeping the thumb-nail (nail-biters cannot become good mathematicians!) on the left-hand corner and stretching the base edge of the paper with the right hand while folding.

FIG. 3. This is a reversed fold for accuracy.

FIG. 4. This shows the first half of the process completed and it is easy to see that the $\triangle ABC$ will be equilateral.

possible, which is a pity as it means that this only of the 5 Platonic solids cannot easily be made in the classroom by the present method."

May I submit the following suggestion which is in the spirit of Mr. Pargeter's 3-plaits if not in the letter. In his note to model 4 he says that having pentagonal faces the dodecahedron cannot be plaited on the lines of the other models. But ten of the twelve faces can! and in the present model the other two faces, a parallel pair, are by the *ad hoc* method suggested for the base of the pyramid and ends of the prism, models 31 & 34. Without a complicated means of preliminary folding, the three strips are best cut separately. They are similar in shape and are cut according to Fig. 5, strip No. 1 being A to N, strip No. 2 A to L and strip No. 3 D to O. $3O$ is now stuck down to $2K$ and $2L$ to $1N$ as shown in Fig. 6. Plaiting then begins by placing $1L$ over $3N$, then $2H$ over $1K$ etc. $1A$ tucks under $1L$ at last, and it is best cut and folded as indicated as it is then less likely to slip. Finally the remaining two faces are formed by tucking the flaps under each other in rotation. This model is firm and as quickly plaited as the other four Platonic solids suggested by Mr. Pargeter.

I encourage as much folded-paper Geometry as I can, since many constructions are more quickly and more logically (and more accurately!) performed and when Mr. Pargeter challenges us to fold an equilateral triangle, I suppose he is thinking of some such method as I referred to above. I would (and do) also use folded pentagons for the development of the dodecahedron.

Chatham Technical School for Boys JAMES BRUNTON

Editorial Note. Mr. A. R. Pargeter made the following invited comment on Mr. Brunton's construction:

I am most interested to see Mr. Brunton's 3-plait for the dodecahedron. When I was first experimenting at extending Gorham's methods, and before I had fully grasped the principles that I set forth in my article, I did make a dodecahedron on a practically identical plan, but I rejected it on account of its lack of symmetry and the number of "ends"; in any case, it was clear that one could not plait a dodecahedron with pentagons only—free vertices would be left flapping, as it were—and I felt sure that there must be something analogous to Gorham's "compound" octahedron. In Mr. Brunton's arrangement the free vertices are all inside the model; and it may in fact be regarded as an ingenious extension of the method I used to construct a triangular prism, model **34** in my article. As beginners find a 6-plait rather tricky to handle, Mr. Brunton's model may well help to fill the gap that I lamented, namely that in the classroom the dodecahedron alone of the 5 Platonic solids was not easily made by plaiting.

FORMULAE FOR VOLUMES, SURFACE AREAS AND RADII OF GYRATION OF SPHERES, ELLIPSOIDS AND SPHEROIDS

By John Satterly

Recently I have been experimenting on the physical properties, dimensions and constants of hens' eggs. During this work I have had necessarily to look up the formulae for areas and volumes of ellipsoids and to invent approximate formulae for use where the correct formulae entails the use of mathematical tables. I now give them to other students.

Table I gives the correct expressions; they will be found in the Textbooks and Mathematical Tables. I have added some approximate expressions which are useful in cases where there is no great inequality in the lengths of the axes; they have the merit of being easily worked out.

Table I

Let a, b, c be the three rectangular semi axial lengths and assume $a > b > c$

	Sphere	Ellipsoid	Prolate Spheroid	Oblate Spheroid
Axes	$a, a, a,$	$a, b, c\ (a > b > c)$	$a, b, b,\ (a > b)$	$a, a, b,\ (a > b)$
Volume, V	$\frac{4}{3}\pi a^3$	$\frac{4}{3}\pi abc$	$\frac{4}{3}\pi ab^2$	$\frac{4}{3}\pi a^2 b$
Surface Area, S	$4\pi a^2$	See below (note 2) $\frac{1}{3}\pi(bc + ca + ab)$†	$2\pi\left(b^2 + \frac{ab}{e}\sin^{-1}e\right)$ $\frac{1}{3}\pi b(2a + b)$†	$2\pi\left(a^2 + \frac{b^2}{2e}\ln\frac{1+e}{1-e}\right)$ $\frac{1}{3}\pi a(a + 2b)$†

Square of radius of gyration or k^2

About which axis?	any axis	Axis a	Axis a	Axis b
Solid body	$\frac{2}{5}a^2$	$\frac{1}{5}(b^2 + c^2)$	$\frac{2}{5}$	$\frac{2}{5}a^2$
Thin shell	$\frac{2}{3}a^2$	$\frac{1}{3}(b^2 + c^2)$* See below†	$\frac{2}{3}b^2$* $\frac{2}{3}b^2\frac{4a + b}{2a + b}$†	$\frac{2}{3}a^2$* $\frac{2}{3}a^2\frac{a + 4b}{a + 2b}$†

Notes (1) The correct formulae are unmarked by * or †. In all cases e is the eccentricity of the ellipse and $= (a^2 - b^2)/a^2$.

(2) The expression for the surface area of an Ellipsoid is complicated. As proved in J. Edwards, Integral Calculus (Macmillan and Co., London and New York., 1921, Vol. 1, p. 842, it is

$$S = 2\pi\left\{c^2 + \frac{ab}{\sin\gamma}[\sin^2\gamma \,.\, E(\gamma, k) + \cos^2\gamma \,.\, F(\gamma, k)]\right\} \qquad (1)$$

where it is assumed $a > b > c$ and

$$\cos\gamma = c/a \qquad \text{and} \qquad k = \frac{a^2}{b^2} : \frac{b^2 - c^2}{a^2 - c^2}$$

and E, F are the appropriate elliptic functions. Here k is not the radius of gyration. The usual books of tables which give elliptic functions (such as J. B. Dale, Five figure tables of Mathematical Functions (Arnold, London), 1932, pp. 68–72) give E and F in terms of θ and ϕ. Here $\theta = \sin^{-1} k$ and ϕ is Edwards' γ.

(3) Edwards also gives as an exercise to be proved: "If a, b, c are very nearly equal the surface area of an ellipsoid is very nearly equal to

$$4\pi a^{2/3}b^{2/3}c^{2/3} \qquad (2)$$

This is easily memorised by noting that the expression $S = 4\pi a^2$ for the sphere is changed for the ellipsoid by distributing the power 2 equally between the axes a, b, c.

Approximate formulae for the surface area of Ellipsoids, marked † in Table I.

All students will have noticed the obvious relation between the expressions for the volume V, and the surface area S of a sphere, namely $V = \frac{4}{3}\pi a^3$, $S = 4\pi a^2$, and $\dfrac{dV}{da} = S$. I have applied similar argument to the ellipsoid. Let us accept the expression $V = \frac{4}{3}\pi abc$ and assume the ellipsoid to expand by a small amount thus adding an ellipsoidal shell to the original ellipsoid. We have

$$dV = \tfrac{4}{3}\pi(da \,.\, bc + a \,.\, db \,.\, c + ab \,.\, dc) \qquad (3)$$

If we make $da = db = dc = t$ this becomes

$$dV = \tfrac{4}{3}\pi t(bc + ca + ab) \qquad (4)$$

and since $S = dV/t$ we have

$$S = \tfrac{4}{3}\pi(bc + ca + ab)$$

For the prolate spheroid

$$c = b \qquad \text{and} \qquad S = \tfrac{4}{3}\pi b(2a + b) \qquad (5)$$

For the oblate spheroid

$$c = a \qquad \text{and} \qquad S = \tfrac{4}{3}\pi a(a + 2b)$$

Table II shows how results by the different methods compare.

TABLE II. Values of surface area, S

Axes	Axes	Correct (Note 2)	Approximate (Note 3)	Approximate marked † in Table I
Ellipsoid a, b, c,	5, 4, 3	199·5	198·6	197·0
Prolate spheroid a, b, b	5,,4, 4	235·3	233·3	234·6
Oblate spheroid a, a, b	5, 5, 4	273·3	270·8	272·5

Moments of Inertia

The *Moment of Inertia* of a body about a given axis is given by $I = \Sigma(mr^2)$ where m is an element of mass of the body at a distance r from the given axis. It is also written as $I = Mk^2$ where $M = \Sigma m$ = the mass of the body and k is the radius of gyration of the body about the given axis. Obviously k is the distance at which if the whole mass were concentrated, the I of the new arrangement would equal the I of the actual body.

Moments of inertia are very useful in Mechanics and Physics and their values for bodies of simple shape are calculated in the textbooks and given in books of tables. It is sufficient to quote k^2.

The k^2's of ellipsoidal shells

We assume the expressions for the k^2's of sphere and the solid ellipsoid and spheroids. They are given in the tables.

Note (4) The expressions marked * for the k^2's of shells are correct if the thickness of the shell at *any* place is proportional to the length of the perpendicular from the centre upon the tangent plane at that place. This is given as an exercise in H. Lamb, Higher Mechanics (Cambridge University Press). 2nd Edit., 1920, p. 71.

A simplified proof is obtained by assuming that the thickness of the shell at the ends of the axes is proportional to the lengths of those axes. We then have $da/a = db/b = dc/c = \lambda$ say

Equation (3) becomes $V = 4\pi\lambda abc$

Let I_a = the moment of inertia of the solid ellipse about the axis a and assume the material of the ellipsoid is of unit density. The textbooks give

$$k_a{}^2 = \tfrac{1}{5}(b^2 + c^2)$$

Therefore:

$$I_a = V \times \tfrac{1}{5}(b^2 + c^2) = \tfrac{4}{3}\pi abc \,.\, \tfrac{1}{5}(b^2 + c^2) = \tfrac{4}{15}\pi(ab^3c + abc^3)$$

$$dI_a = \tfrac{4}{15}\pi\{(\lambda ab^3c + a \,.\, 3b^2\lambda b \,.\, c + ab^3\lambda c) + (\lambda abc^3 + a\lambda bc^3 + ab \,.\, 3c^2\lambda c)\}$$

$$= \tfrac{4}{15}\pi\lambda abc(5b^2 + 5c^2) = \tfrac{4}{3}\pi\lambda abc(b^2 + c^2)$$

$$\left.\begin{aligned} &\text{The } k_a^2 \text{ of the shell} = \frac{\text{Increase of } I_a}{\text{Increase of } V} = \frac{b^2 + c^2}{3} \\ &\text{With the same assumption} \\ &\quad k_a^2 \text{ of a prolate spheroidal shell of axes } a, b, b = 2b^2/3 \\ &\quad \text{and } k_b^2 \text{ of an oblate spheroidal shell of axes } a, a, b = 2a^2/3 \end{aligned}\right\} \quad (6)$$

Approximate expressions for k^2 of shells (marked † in Table I)

As above we assume the shell to be of equal thickness t at the ends of the axes and $dV = \frac{4}{3}\pi t(bc + ca + ab)$, see Equation 4. As before $I_a = \frac{4}{15}\pi(ab^3c + abc^3)$

$$\therefore \quad dI_a = \tfrac{4}{15}\pi t\{(b^3c + 3ab^2c + ab^3) + (bc^3 + ac^3 + 3abc^2)\}$$
$$= \tfrac{4}{15}\pi t\{a(b^3 + c^3) + bc(b^2 + c^2) + 3abc(b + c)\}$$

Divide this by dV and we get

$$k_a^2 = \frac{1}{5}\frac{a(b^3 + c^3) + bc(b^2 + c^2) + 3abc(b + c)}{bc + ca + ab} \quad (7)$$
$$= \frac{1}{5}\frac{a(b + c)^3 + bc(b^2 + c^2)}{bc + ca + ab}$$

If the ellipsoidal shell is a spherical shell of radius a

$$k_a^2 = \tfrac{2}{3}a^2.$$

For the prolate spheroid a, b, b put $b = c$ and $a > b$ and

$$k_a^2 = \tfrac{2}{3}b^2\frac{4a + b}{2a + b} \quad (8)$$

For the oblate spheroidal shell b, a, a replace a in the expression by b and both b and c by a and

$$k_b^2 = \tfrac{2}{3}a^2\frac{a + 4b}{a + 2b} \quad (9)$$

To get k_b^2 of a prolate spheroidal shell (a, b, b) replace a by b and c by a and we get

$$k_b^2 = \frac{1}{5}\frac{(b + a)^3 + a(b^2 + a^2)}{b + 2a} = \frac{1}{5}\frac{b^3 + 4b^2a + 3ba^2 + 2a^3}{b + 2a} \quad (10)$$

A correct formula for the k_a^2 of a thin prolate spheroidal shell (a, b, b) of uniform thickness all over

Professor C. Barnes, a mathematical colleague, worked out for me the value of k_a^2 for a thin prolate spheroidal shell of small *uniform*

thickness. It is

$$k_a^2 = \frac{b^2}{e^2} \cdot \frac{F - (1 - e^2)G}{G} . \qquad (10)$$

where $G = \frac{1}{2}\{e(1 - e^2)^{1/2} + \sin^{-1} e\}$

and

$$F = \tfrac{1}{4}\{e(1 - e^2)^{3/2} + \tfrac{3}{2}e(1 - e^2)^{1/2} + \tfrac{3}{2}\sin^{-1} e\}$$
$$= \tfrac{1}{4}\{e(1 - e^2)^{3/2} + 3G\}$$

Table III gives numerical values for prolate spheroidal shells for a wide range of eccentricities. Note that the more nearly the spheroid is a sphere the more correct are the approximate formulae.

TABLE III
Prolate spheroidal shells

Dimensions		Eccentricity		Area		k_a^2 depends on assumption		
a	b	e^2	e	Correct, see Table I	By Eq. 5	Correct by Eq. 10	Approximate by Eq. 8	Correct by Eq. 6
100	1	0·999	0·99995	—	—	0·75	0·80	0·67
10	1	0·9900	0·9950	99·10	87·97	0·748	0·78	0·67
13	5	0·8521	0·9231	677·5	649·2	18·17	18·4	16·7
5	3	0·6400	0·8000	164·6	163·4	6·35	6·37	6·0
3·03	2·07	0·5333	0·7303	71·09	70·50	2·98	2·98	2·85
3·27	2·29	0·5096	0·7130	85·34	84·70	3·65	3·65	3·49
5	4	0·3600	0·6000	235·3	234·6	10·96	10·97	10·67
13	12	0·1479	0·3846	1911	1910	97·01	97·01	96·0

The 3·03, 2·07 and the 3·27, 2·29 are the mean dimensions in cms of two very nearly symmetrical egg shells of the domestic hen and the great blue heron respectively. By experiments with a torsion pendulum the experimental k_a^2's of these were 2.99 and 3·62 cm^2 respectively and the k_b^2's were 4·26 and 4·92 cm^2 respectively. For the latter the k_b^2's calculated by Equation 10 were 4·26 and 5·07 cm^2 respectively. The thickness of the hen's egg shell is about 0·037 cm of which the inside shell membrane is responsible for about 0·003 cm. Other egg shells gave similar results. Of course egg shells which are symmetrical are not necessarily prolate spheroids.

J. S.

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ON THE NOTATION FOR PARTIAL DERIVATIVES

H. A. THURSTON

Theorem: $0 = 1$.

Proof: If $\phi(x, y) = x^2 - y^2$ (for all x and y), then $\phi_y(x, y) = -2y$ (for all x and y). Similarly, if $f(p, z) = p^2 - z^2$ then $f_z(p, z) = -2z$. And if $F(z, p) = z^2 - p^2$, then $F_p(z, p) = -2p$. Also $F_z(z, p) = 2z$.

In particular,

$$f_z(z, z) = -2z$$

and

$$F_z(z, z) = 2z$$

for every z.

But ϕ, f and F are all the same function. Therefore f_z and F_z are the same function. Therefore $-2z = 2z$ for every z, whence easily $0 = 1$.

Q.E.D.

I have been using the definition:—

(i)
$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h};$$
$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}.$$

This is common in text-books of calculus. So is:—

(ii)
$$f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h};$$
$$f_y(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h}.$$

Definitions (i) and (ii) are not always recognized as different; but if we apply (ii) to the functions defined in the theorem we get

$\phi_y(x, y) = -2y$, $f_y(p, z) = -2z$, $F_y(z, p) = -2p$, and $F_x(z, p) = 2z$.

The reader may object that these formulae are absurd; I shall not deny this, but shall insist that they follow from the definition.

The familiar formula for differentiating a function of two functions is

(iii)
$$f_x(u, v) = f_u(u, v) \cdot \frac{\partial u}{\partial x} + f_v(u, v) \cdot \frac{\partial v}{\partial x}.$$

There is a similar one for f_y. The formulae cannot be proved from definition (i), because this fails to define $f_x(u, v)$. Can it be proved

from definition (ii)? Does (ii) define f_u? Because nothing is said to the contrary, we should expect (ii) to hold "for all x and y" (strictly: for all x, y, a, b and f for which the limits exist): i.e. x and y are free variables, like a and b, and like the x and y in (i), and indeed like f, except that f is not a number-variable but a function-variable. But then f_u is the same as f_x; so are f_r, f_k, f_z, ... even f_y. And now the two parts of (ii) contradict each other.

The fact is that (i) and (ii) are intrinsically unsound: (i) because the symbol x plays two different rôles; (ii) because there is a free variable in the definiendum which does not occur in the definiens. They can both be put right if we realize that the suffix should be not a variable but a marker to show which of the arguments of f differs in the two occurrences of f in the definiens. Using the obvious markers 1 and 2 (as Landau and Osgood have done)—we get:—

(iv)
$$f_1(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h};$$
$$f_2(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

(Whether we use a and b or x and y to denote the arguments matters not a jot).

With the functions defined in the opening theorem we now get

$$\phi_2(x, y) = -2y, f_2(p, z) = -2z, F_2(z, p) = -2p, \text{ and } F_1(z, p) = 2z.$$

These are neither absurd nor inconsistent with $\phi = f = F$.

Formula (iii) follows (for well-behaved functions) in the form:—

(v)
$$\begin{cases} \text{If} \quad \psi(x, y) = \phi(\alpha(x, y), \beta(x, y)), \text{ then} \\ \psi_1(x, y) = \phi_1(\alpha(x, y), \beta(x, y)) \cdot \alpha_1(x, y) \\ \qquad\qquad + \phi_2(\alpha(x, y), \beta(x, y)) \cdot \beta_1(x, y) \end{cases}$$

(With care this can be abbreviated to

$$\frac{\partial}{\partial x} \phi(u, v) = \phi_1(u, v) \cdot \frac{\partial u}{\partial x} + \phi_2(u, v) \cdot \frac{\partial v}{\partial x}.$$

It is possible to produce a definition which uses formula (ii) and which is not unsound:— for every function f and number-pair (a, b) for which the relevant limits exist and for the *particular* symbols x, y, the expressions $f_x(a, b)$ and $f_y(a, b)$ are defined by (ii). It follows, of course, that the functions f_x and f_y are defined.

But although this definition escapes the troubles I have mentioned, it has some disadvantages of its own. It fails for instance, to define $f_u(u, v)$ or $g_t(a, b)$ or any such formula in which the suffix is neither

x nor y. The reason why this is a disadvantage is that most standard text-books (e.g. Goursat, Courant, Hardy) sometimes use letters other than x and y as suffixes. Further, it becomes rather unnatural when applied to geometry. If we have a point-function expressed in polar coordinates: e.g. $f(r, \theta) = r \cdot \cos\theta$ then if we differentiate with respect to r *using this definition*, we get $f_x(r, \theta) = \cos\theta$. This is not merely not standard usage; it actually conflicts with standard usage, in which (for this particular f) $f_x(r, \theta)$ would denote $\frac{\partial}{\partial x}(r \cdot \cos\theta)$ i.e. $\frac{\partial}{\partial x} x$ which equals 1. So this usage, though theoretically sound, is scarcely satisfactory.

This usage, in fact, comes down to *realizing* that the suffixes are "markers", not variables, but nevertheless continuing to use letters for them. But if one does realize that the suffixes are markers, it seems in every way preferable to take one more step, and use numbers as suffixes. And once more, when we do this, the difficulties disappear; there is no conflict with standard usage, and one does not have the quaint device of denoting a differentiation with respect to r by a suffix x.

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GLEANINGS FAR AND NEAR

1936. Mathematical recreations in the eighteenth century:—Whilst I was Professor of Chemistry, I dissected a subject which I had procured from London, in order to perfect myself in Anatomy; my laboratory was my theatre, and Professor Waring, known to Europe by his mathematical publications, and my old friend Preston (afterwards Bishop of Ferns) were my assistants. When we had finished the business we put what remained of the body into a box, and commissioned an old soldier to bury it in the fields. The man thought the box was worth something, and instead of burying it he opened it, and poured the contents into the Cam, and as there happened then to be a great flood, some of them were drifted on shore and excited a great suspicion of murder having been committed; but as no person was either taken up or suspected of it, we carefully kept our secret, and thus probably escaped being stoned, like anatomists of old, by a superstitious populace. *Anecdotes of the life of Richard Watson*, I., (1818), pp. 237–8. [Per Dr. G. N. Watson. Preston and Watson were both second wranglers, in 1753 and 1759, the former beating a future Master of Trinity, Postlethwaite, by one place, and both were subsequently Fellows of Trinity. Waring, a Magdalene man, was Senior in 1757; he also graduated M.D. in 1767 and, while he migrated to Trinity, he was never a Fellow of that House, nor does his name appear in the Admission Books.]

SENTENCE LOGIC AS AN INTRODUCTION TO AXIOMATIC SYSTEMS

By J. Hooley

"The moral of this story seems to be that mathematical reasoning based on an axiomatic basis is not difficult, ... " "Modern mathematics is turning more and more to axiomatic methods, and there is a great deal to be said for introducing students to them at an early stage, when, I believe, they would take to them more readily, and enjoy them."

These are quotations from Professor Hodge's Presidential Address to the Association in 1955 (*Gazette Vol. XXXIX, No:* 329, *Sept.* 1955). The axiomatic system Professor Hodge had in mind was, of course, axiomatic projective geometry, but the present article aims to give an account of an axiomatic system whose intuitive interpretation is presupposed and informally used in axiomatic projective geometry. This system is the two-valued classical sentence calculus. What makes this calculus more instructive for the beginner than geometry is its amenability to metamathematical study, since it may so easily be shown to be consistent, complete and decidable.

In any formal system, the "statements" that are written down are entirely made up of signs taken from certain prescribed lists, and made up only according to prescribed rules. In the sentence calculus, there are, among the signs, variables and constants, the former becoming propositions (i.e. sentences which declare that such-and-such is so) in the interpretation; and the latter becoming the means of taking these propositions in combination with each other (or of denying a proposition) to form composite propositions. For example the constant $\vee$ corresponds to the word "or" in the interpretation, $\&$ to the word "and", $\rightarrow$ to "implies" and $\neg$ to "not". We shall not regard $\&$ and $\rightarrow$ as signs of the system but only as abbreviations for certain combinations of $\vee$ and $\neg$.

The rules that we shall give for building sentences from signs were chosen of course with one eye on the interpretation, in such a way as to yield only meaningful statements. The names "not", "or" etc., may still be used in reading the signs $\neg$, $\vee$, ..., but it is preferable, at first anyway, not to be reminded too strongly of the interpretation, and to use some other names, for instance "vee", "barb" ($\vee$, $\neg$) etc.

The axioms are now nothing more than certain initial combinations of signs.

As in mathematics, we shall call the formal sentences which we prove "theorems"; but having been careful to say first what a

sentence is, and precisely which sentences form the starting point of the investigations, we shall have to be equally careful to say what devices shall be at our disposal for carrying out the steps of a proof, and of what this proof shall consist. The operations of the calculus become entirely automatic, and ask no more of the reader than the ability to recognise signs and substitute one sign for another.

Now to describe the calculus:

as variables we shall use small italic letters from p onwards in the alphabet, i.e. p, q, r, ...; the constants are $\vee$ and $\neg$. These are all the signs of the calculus and the following four rules give a complete account of what is meant by a sentence:

(α) a variable is a sentence;

(β) if A stands for any sentence, $\neg(A)$ stands for a sentence;

(γ) if A and B stand for any sentences, $(A) \vee (B)$ stands for a sentence;

(δ) the only sentences are those constructed according to (α), (β), (γ).

The brackets in (β) and (γ) are not signs of the calculus but serve only to define to which formal expressions the $\neg$ or $\vee$ is applied. The capital letters A, B are likewise not signs of the formal calculus but enable us to talk about sentences in general.

Guided by the familiar usage which takes "A implies B" to mean "B is true or A is false", we write $A \rightarrow B$ as an abbreviation for $\neg A \vee B$. In a similar way we may justify writing $A \,\&\, B$ for $\neg(\neg A \vee \neg B)$.

The following four sentences are the axioms:

(a) $(p \vee p) \rightarrow p$ (which, unabbreviated is $\neg(p \vee p) \vee p$);

(b) $p \rightarrow (p \vee q)$;

(c) $p \vee q \rightarrow q \vee p$;

(d) $(p \rightarrow q) \rightarrow [(r \vee p) \rightarrow (r \vee q)]$.

We can easily verify that each of these is indeed a sentence: (a) for example is constructed as in (γ) with $\neg(p \vee p)$, p as the A, B; these constituents are sentences, for by (α), p is a sentence, then by (γ) so is $p \vee p$, and by (β) so is $\neg(p \vee p)$. Brackets, where cumbersome, will be dropped in accordance with the convention that the constants and abbreviations bind tightest in the following order: $\neg$, $\vee$, $\&$, $\rightarrow$, just as $\cdot$ binds more tightly than $+$ in elementary algebra. Thus (a) could be written $p \vee p \rightarrow p$ without fear of its being mistaken for $p \vee (p \rightarrow p)$.

Next for consideration are the rules of procedure—the "moves" of the "game"—which allow us to make "inferences". There are two of these:

(i) the rule of substitution by means of which we get a new proved sentence (theorem), S, from one, S^*, already proved, by

substituting any sentence T whatever, for all occurrences in S^* of any one of its variables; and

(ii) the rule of inference by *modus ponens* which allows us to infer the sentence T from already proved sentences S and $S \rightarrow T$. A *proof* is then merely a finite list of sentences each of which follows by either of rules (i) and (ii) from sentences higher up the list, or is an axiom. A proof is a proof *of* its final sentence. Proofs may be abbreviated by deriving further rules of inference from our primitive rules (i) and (ii). The derivation of such rules and the proofs of some theorems will now follow, providing examples of the workings of the calculus.

(iii) If, for some sentence A, $A \vee A$ is a theorem, then so is A: for having proved $A \vee A$ we can continue thus

(1)	$A \vee A$	(just proved)
(2)	$p \vee p \rightarrow p$	(ax. (a))
(3)	$A \vee A \rightarrow A$	(rule (i) on (2), with ax. (a) as the S^*, A as the T and (3) as the S)
(4)	A	(rule (ii) on (1), (3), with $A \vee A$ as the S, and A as the T)

(iv) If A is a theorem and B any sentence, then $A \vee B$ is a theorem (prove this using ax. (b)).

(v) If $A \vee B$ is a theorem, so is $B \vee A$ (use ax. (c)).

(vi) Similarly from a theorem $A \rightarrow B$ can be derived another $C \vee A \rightarrow C \vee B$, where C is any sentence.

(vii) *Theorem 1:* $(p \rightarrow q) \rightarrow [(r \rightarrow p) \rightarrow (r \rightarrow q)]$

Proof:

(1)	$(p \rightarrow q) \rightarrow [(r \vee p) \rightarrow (r \vee q)]$	(ax. (d))
(2)	$(p \rightarrow q) \rightarrow [(\neg r \vee p) \rightarrow (\neg r \vee q)]$	(subst. of $\neg r$ for r by (i) in (1))
(3)	$(p \rightarrow q) \rightarrow [(r \rightarrow p) \rightarrow (r \rightarrow q)]$	((2) abbreviated)

(viii) If $A \rightarrow B$ and $B \rightarrow C$ are theorems, then $A \rightarrow C$ is a theorem, for:

(1)	$(p \rightarrow q) \rightarrow [(r \rightarrow p) \rightarrow (r \rightarrow q)]$	(Thm. 1)
(2)	$(B \rightarrow C) \rightarrow [(A \rightarrow B) \rightarrow (A \rightarrow C)]$	(rule (i) three times on (1))
(3)	$B \rightarrow C$	(a theorem)
(4)	$(A \rightarrow B) \rightarrow (A \rightarrow C)$	(rule (ii) on (3), (2))
(5)	$A \rightarrow B$	(a theorem)
(6)	$A \rightarrow C$	(rule (ii) on (5), (4))

(ix) *Theorem 2:* $\neg p \vee p$

Proof: (1) $p \to p \vee q$ (ax. (b))
(2) $p \to p \vee p$ (rule (i) on (1))
(3) $p \vee p \to p$ (ax. (a))
(4) $p \to p$ (rule (viii) on (2), (3))
(5) $\neg p \vee p$ ((4) unabbreviated)

(x) *Theorem 3:* $p \vee \neg p$ (Use rule (v) on Thm. 2.)

(xi) *Theorem 4:* $p \to \neg \neg p$ (Substitute $\neg p$ for p in Thm. 3 and abbreviate.)

(xii) *Theorem 5:* $(q \to p) \to (\neg p \to \neg q)$

Proof: (1) $p \to \neg \neg p$ (Thm. 4)
(2) $\neg q \vee p \to \neg q \vee \neg \neg p$ (rule (vi) on (1))
(3) $\neg q \vee \neg \neg p \to \neg \neg p \vee \neg q$ (from ax. (c) by rule (i))
(4) $\neg q \vee p \to \neg \neg p \vee \neg q$ (rule (viii) on (2), (3))
(5) $(q \to p) \to (\neg p \to \neg q)$ ((4) abbreviated)

(xiii) If $A \to B$ is a theorem, then so is $\neg B \to \neg A$. (Prove this by substitution in Theorem 5, then use rule (ii).)

(xiv) *Theorem 6:* $(p \to q) \to [p \vee r \to q \vee r]$

Shortened Proof:

(1) $r \vee q \to q \vee r$ (rule (i) on ax. (c))
(2) $\neg (r \vee p) \vee (r \vee q) \to \neg (r \vee p) \vee (q \vee r)$ (rule (vi) on (1))
(3) $\neg (r \vee p) \vee (r \vee q) \to (q \vee r) \vee \neg (r \vee p)$ (subst. in ax. (c) then rule (viii))
(4) $\neg (r \vee p) \to \neg (p \vee r)$ (from ax. (c) and rule (xiii))
(5) $(q \vee r) \vee \neg (r \vee p) \to (q \vee r) \vee \neg (p \vee r)$ (rule (vi) on (4))
(6) $\neg (r \vee p) \vee (r \vee q) \to (q \vee r) \vee \neg (p \vee r)$ (from (3), (5))
(7) $\neg (r \vee p) \vee (r \vee q) \to \neg (p \vee r) \vee (q \vee r)$ (ax. (c); rule (viii))
(8) $[(r \vee p) \to (r \vee q)] \to [(p \vee r) \to (q \vee r)]$ (meaning of $\to$)
(9) $(p \to q) \to [(p \vee r) \to (q \vee r)]$ (ax. (d), (8) and rule (viii))

(xv) *Theorem 7:* $(p \to q) \to [(q \to r) \to (p \to r)]$.

Prove this by substitutions in Theorems 5 and 6, then link the results by rule (viii).

(xvi) *Theorem 8:* $[p \to (p \to q)] \to (p \to q)$

Proof:

(1)	$r \to r \vee q$	(from ax. (b))
(2)	$(r \to r \vee q) \to [r \vee (r \vee q) \to (r \vee q) \vee (r \vee q)]$	(from Thm. 6)
(3)	$r \vee (r \vee q) \to (r \vee q) \vee (r \vee q)$	(rule (ii) on (1), (2))
(4)	$(r \vee q) \vee (r \vee q) \to r \vee q$	(from ax. (a))
(5)	$r \vee (r \vee q) \to r \vee q$	(rule (viii) on (3), (4))
(6)	$[p \to (p \to q)] \to (p \to q)$	(subst. of $\neg p$ for r in (5))

(xvii) *Theorem 9:* $(p \vee q) \vee r \to p \vee (q \vee r)$

Proof:

(1)	$q \to q \vee r$	(from ax. (b) and rule (i))
(2)	$p \vee q \to p \vee (q \vee r)$	(rule (vi) on (1))
(3)	$(p \vee q) \vee r \to [p \vee (q \vee r)] \vee r$	(from (2) by Thm. 6 and rule (ii))
(4)	$r \to q \vee r$	(from axs. (b), (c))
(5)	$q \vee r \to p \vee (q \vee r)$	(from axs. (b), (c))
(6)	$r \to p \vee (q \vee r)$	(from (4), (5))
(7)	$[p \vee (q \vee r)] \vee r \to [p \vee (q \vee r)] \vee [p \vee (q \vee r)]$	(rule (vi) on (6))
(8)	$[p \vee (q \vee r)] \vee r \to [p \vee (q \vee r)]$	(from (7) using ax. (a))
(9)	$(p \vee q) \vee r \to p \vee (q \vee r)$	(rule (viii) on (3), (8))

At this point we make certain observations. The derived rules (iii), (iv), etc., are not (formal) theorems, being sentences in English not in the formal calculus; however, in informal mathematics we probably should call them "theorems". Such informal theorems are part of the external discussion of, and reasoning about the formal system and are called *metamathematical theorems* (*meta-theorems*); their proofs incorporate snatches of quasi-formal proof involving the *metamathematical letters* $A, B, C, \ldots$, whose rôle and status have already been described.

Now look at the working in (viii): if $A \to B$ were not a theorem but were still introduced at step (5), we should nevertheless get $A \to C$ at step (6). $A \to C$ would not now be a theorem necessarily, but merely a consequence of *assuming* $A \to B$ as an *hypothesis*, i.e. of adding $A \to B$ to the list of axioms. In such a case we write symbolically $A \to B \vdash A \to C$. ($B \to C$ is still a theorem and we could write $\vdash B \to C$ to express this.)

The symbol $\vdash$ may be read as follows:

for $D \vdash E$ read "D yields E", and

for $\vdash E$ read "E is provable" or "E is a theorem";

it is a metamathematical and not a formal symbol.

If $D \vdash E$, we say that E is *deducible* from D.

We shall now give a proof of one of the most important metatheorems connected with the sentence calculus, the Deduction Theorem. Let Γ represent a finite (possibly empty) sequence of sentences and let A represent a sentence. Then the Theorem states that if with Γ, A as hypotheses, we can deduce a sentence B (in symbols $\Gamma, A \vdash B$), then from the hypotheses Γ alone, we can deduce $A \to B$, provided that rule (i) is not applied (on any variable occurring in A) to a sentence depending on A by rules (i) and (ii).

Proof by induction on the length l of (number l of sentences in) the deduction of B from Γ, A.

If $l = 1$ then B must be an axiom, A, or a member of Γ.

(a) If B is an axiom:

(1)	$\vdash B$	(an axiom)
(2)	$\vdash B \to (A \to B)$	($p \to (q \to p)$ is a formal theorem—prove it!)
(3)	$\vdash A \to B$	(rule (ii) on (1), (2))

Hence in a vacuous sort of way, $\Gamma \vdash A \to B$.

(b) If B is A:

(1)	$\vdash A \to A$	($p \to p$ is a theorem)
(2)	$\vdash A \to B$	(B is A)

Hence again $\Gamma \vdash A \to B$.

(c) If B is a member of Γ prefix each $\vdash$ of (a) by Γ.

As hypothesis of induction, assume the theorem to hold for $l \leqslant k$, and the deduction $\Gamma, A \vdash B$ to have length $k + 1$. Then B must follow from an earlier sentence B^* by rule (i), or else B follows from two earlier sentences of the form P, $P \to B$ by rule (ii), but in any case the lengths of the deductions of B^* on the one hand, or P and $P \to B$ on the other, are less than $k + 1$ and so we have by hypothesis, deductions of $A \to B^*$ in the first case and of $A \to P$ and $A \to (P \to B)$ in the second, from Γ alone.

Case 1. B follows from B^*.

If B^* does not depend upon A:

$\Gamma \vdash B^* \vdash B$ continue as in (c) above.

If B^* does depend upon A then the variables of B^* on which substitution is made, do not occur in A otherwise the proviso made in stating the theorem would not be preserved. So

(1) $\Gamma, A \vdash B^*$ (by a deduction of length $\leqslant k$)
(2) $\Gamma \vdash A \to B^*$ (induction hypothesis)
(3) $\Gamma \vdash A \to B$ (rule (i) on $A \to B^*$)

Case 2. B follows from P, $P \to B$.

(1) $\Gamma \vdash A \to P$ (induction hypothesis)
(2) $\vdash (A \to P) \to [(P \to B) \to (A \to B)]$ (from Thm. 7)
(3) $\Gamma \vdash (P \to B) \to (A \to B)$ (rule (ii) on (1), (2))
(4) $\vdash [(P \to B) \to (A \to B)] \to [\{A \to (P \to B)\} \to \{A \to (A \to B)\}]$ (from Thm. 1)
(5) $\Gamma \vdash \{A \to (P \to B)\} \to \{A \to (A \to B)\}$ (rule (ii))
(6) $\Gamma \vdash A \to (P \to B)$ (induction hypothesis)
(7) $\Gamma \vdash A \to (A \to B)$ (rule (ii))
(8) $\vdash [A \to (A \to B)] \to (A \to B)$ (from Thm. 8)
(9) $\Gamma \vdash A \to B$ (rule (ii))

This completes the proof of the deduction theorem.

An important thing about this theorem is that it forms the basis of an alternative formulation of the sentence calculus, called *natural inference*, whose virtues are that it is quicker in action than the old formulation and that it resembles ordinary reasoning much more closely. It is not axiomatic but formal nevertheless, being based on a set of eight rules of inference for the introduction and elimination of the signs $\neg$, $\vee$, $\to$, $\&$. The rules give exactly the same set of provable sentences as do axioms (a)–(d) with rules (i) and (ii) and so this system is fully equivalent to the axiomatic one.

Here are the rules set out in a table:

	Intro.	Elim.
$\neg$	If $A \vdash B$ and $A \vdash \neg B$ then $\vdash \neg A$	$\neg \neg A \vdash A$
$\vee$	$A \vdash A \vee B$ $B \vdash A \vee B$	If $A \vdash C$ and $B \vdash C$ then $A \vee B \vdash C$
$\to$	If $A \vdash B$ then $\vdash A \to B$	If $\vdash A$ and $\vdash A \to B$, then $\vdash B$
$\&$	If $\vdash A$ and $\vdash B$ then $\vdash A \ \& \ B$	$A \ \& \ B \vdash A$ $A \ \& \ B \vdash B$

A, B, C standing for any sentences.

On the basis of this formulation, many interesting properties of the sentence calculus can be derived. Firstly however, let us

introduce another abbreviation, $\leftrightarrow$; $A \leftrightarrow B$ shall mean $(A \rightarrow B)$ $\&$ $(B \rightarrow A)$, i.e. $\neg[\neg(\neg A \vee B) \vee \neg(\neg B \vee A)]$; its interpreted reading is "if and only if", and it expresses logical equivalence. $\&$, $\vee$ and $\leftrightarrow$ have properties similar to those of $+$, $\cdot$ and $=$ in arithmetic. In particular $\&$ and $\vee$ are commutative and associative, but unlike $+$ and $\cdot$, *each* is distributive over the other. These and some other equivalences are given below.

(1) $\vdash (p \vee q) \leftrightarrow (q \vee p)$; $\vdash (p \,\&\, q) \leftrightarrow (q \,\&\, p)$;

(2) $\vdash (p \vee q) \vee r \leftrightarrow p \vee (q \vee r)$; $\vdash (p \,\&\, q) \,\&\, r \leftrightarrow p \,\&\, (q \,\&\, r)$;

(3) $\vdash p \vee (q \,\&\, r) \leftrightarrow p \vee q \,\&\, p \vee r$;
$\vdash p \,\&\, (q \vee r) \leftrightarrow (p \,\&\, q) \vee (p \,\&\, r)$;

(4) $\vdash p \,\&\, q \leftrightarrow \neg(\neg p \vee \neg q)$; $\vdash p \vee q \leftrightarrow \neg(\neg p \,\&\, \neg q)$;

(5) $\vdash p \vee p \leftrightarrow p$; $\vdash p \,\&\, p \leftrightarrow p$;

(6) $\vdash \neg\neg p \leftrightarrow p$.

Furthermore one can show that if $\vdash A \leftrightarrow B$ and if A occurs as part of a sentence C, then $\vdash C \leftrightarrow C^*$, where C^* arises from C by replacing A by B at any occurrence(s) of A in C. We can now reduce any sentence S, in n distinct variables, by a chain of equivalences ($\leftrightarrow$ is transitive, of course) to a *normal form* (the *conjunctive* normal form), which looks like this: $A_1 \,\&\, A_2 \,\&\, \ldots \,\&\, A_k$, where each A_i looks like this: $B_1 \vee B_2 \vee \ldots \vee B_m$, where each B_j is either a variable or the negation of a variable, and every variable of S occurs in every A_i either negated, unnegated or both (N.B., $k \leqslant 3^n$, $n \leqslant m \leqslant 2n$). The reduction is effected as follows: occurrences of $\rightarrow$ and $\leftrightarrow$ in S are expressed in terms of $\vee$, $\&$ and $\neg$; by (4) and (6), $\neg$ is made to apply directly to variables only (double $\neg$'s are suppressed immediately they occur, by (6)); then, distributing $\vee$ over $\&$, i.e. "multiplying-out", something like normal form is reached. To complete the process, repetitions of a B_j in the same A_i can be suppressed—(5)—leaving no more than one in each; and a missing variable (p, say) of S can be provided in any A_i which lacks it, by replacing A_i by $A_i \vee (p \,\&\, \neg p)$ (since $\vdash A_i \leftrightarrow A_i \vee (p \,\&\, \neg p)$) and finally "multiplying-out" again. As an example, axiom (d) reduces thus:

$$\vdash (p \rightarrow q) \rightarrow (r \vee p \rightarrow r \vee q) \leftrightarrow \neg(\neg p \vee q) \vee [\neg(r \vee p) \vee (r \vee q)]$$
$$\leftrightarrow (p \,\&\, \neg q) \vee [(\neg r \,\&\, \neg p) \vee r \vee q]$$
$$\leftrightarrow (p \,\&\, \neg q) \vee [(\neg r \vee r \vee q) \,\&\, (\neg p \vee r \vee q)]$$
$$\leftrightarrow (p \vee r \vee \neg r \vee q) \,\&\, (\neg q \vee r \vee \neg r \vee q)$$
$$\&\, (p \vee \neg p \vee r \vee q) \,\&\, (\neg q \vee \neg p \vee r \vee q)$$

We shall use this normal form subsequently.

As we observed earlier, the sentence calculus is consistent, complete and decidable, and we now consider what this means. We say that the calculus is consistent if for any sentence A, at most one

of A and $\neg A$ is a theorem. There is another definition of consistency which, for the sentence calculus and systems having certain properties in common with it, is equivalent to the former; it is that the system is consistent if it has at least one unprovable sentence. We can show this as follows.

1. If the system is consistent in the first sense $p \,\&\, \neg p$ is not provable, otherwise $\&$-elim. would give $\vdash p$ and $\vdash \neg p$.

2. If the calculus is inconsistent in the first sense, let A be a sentence such that $\vdash A$ and $\vdash \neg A$, and let B be *any* other sentence, then, using chains of $\vdash$'s in an obvious way,

$$A, \neg A, \neg B \vdash \neg A; \quad A, \neg A \vdash \neg B \to \neg A \vdash A \to B;$$

but $A, \neg A \vdash A$ whence $A, \neg A \vdash B$; but $\vdash A$ and $\vdash \neg A$ and so $\vdash B$, i.e. every sentence is provable, which means that if there is a sentence which is not provable then the calculus is consistent in the first sense.

We shall prove consistency by means of a *valuation*: this is a calculation process in which each variable ranges over a domain $\{T, F\}$ and each constant is interpreted as an arithmetical function over this domain taking T and F as values (the letters T, F suggest "true" and "false", of course). These functions we define by means of the tables set out below:

A	B	$\neg A$	$A \vee B$	$A \,\&\, B$	$A \to B$	$A \leftrightarrow B$
T	T	F	T	T	T	T
T	F	F	T	F	F	F
F	T	T	T	F	T	F
F	F	T	F	F	T	T

The "truth value" of any sentence can now be computed by applying these tables for a given assignment of T's and F's to the variables in the sentence, e.g. let us evaluate

$$\{(p \vee q) \,\&\, (p \to r) \,\&\, (q \to r)\} \to r$$

for all 8 $(= 2^3)$ possible assignments of values to p, q, r.

p	q	r	$p \vee q$	$p \to r$	$q \to r$	$(p \vee q) \,\&\, (p \to r) \,\&\, (q \to r)$	$S \to r$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	T	T	T	T	T
T	F	F	T	F	T	F	T
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	T
F	F	T	F	T	T	F	T
F	F	F	F	T	T	F	T

where S is the sentence at the head of column seven.

We note that for all assignments of T's and F's to p, q, r *this* sentence gets the value T, and this is the property that we want in our consistency proof. For, firstly if a sentence A has the property then the final value column in the table for $\neg A$ will consist entirely of F's, thus if we can show that all provable sentences always get the value T, the job is done. The reader will easily verify that all four axioms are, as we say, *logically true*, i.e. take only the value T. Substitution of a sentence for a variable in a logically true (l.t.) sentence obviously gives an l.t. sentence. If A and $A \rightarrow B$ are both l.t. then B must have the value T too, which means that rule (ii) leads only to l.t. sentences when applied to such. But all provable sentences arise from the axioms by applications of rules (i) and (ii), hence they are all l.t. and the calculus is consistent. Notice however that we have not proved that all l.t. sentences are provable: if the calculus has this property, we say that it is *complete* (with respect to the truth tables). The sentence calculus in fact does have this property, for let C be an l.t. sentence and let C' be its equivalent in conjunctive normal form. Since $\vdash C \leftrightarrow C', C \leftrightarrow C'$ is l.t. and so C' is l.t. (see table for $\leftrightarrow$). Now, using our previous notations, each A_i (in C') must contain some variable together with its negation; i.e. it must look like this $B_1 \vee B_2 \vee \ldots \vee p_i \vee \neg p_i \vee \ldots \vee B_m$, otherwise an appropriate assignment to the variables (T to each negated and F to each unnegated variable in such an A_i) would reduce that A_i, and thence the whole conjunction $A_1 \,\&\, \ldots \,\&\, A_k$, to F, which under our hypotheses cannot be. But, $\vdash p_i \vee \neg p_i$ (from Thm. 3) whence by $\vee$-intro., $\vdash B_1 \vee \ldots \vee p_i \vee \neg p_i \vee \ldots \vee B_m$, that is $\vdash A_i$ for each i ($i = 1, \ldots, k$); then by $\&$-intro., $\vdash A_1 \,\&\, A_2 \,\&\, \ldots \,\&\, A_k$, that is $\vdash C'$. Finally by reversing the chain of equivalences leading from C to C', we conclude $\vdash C$, and completeness is proved. Furthermore, the calculus is complete in another sense: might it not be possible to enlarge the system, to get more theorems from it simply by adding another sentence as axiom (e)? If this new axiom is deducible from the former axioms, the system is not enlarged by its introduction; if it is not so deducible, then it is not l.t., and by a careful assignment of values to its variables, could be given the value F. If we now applied rule (i) by substituting $p \vee \neg p$ for each variable with the value T (under the above assignment) and $p \,\&\, \neg p$ for each with the value F, the resulting sentence (P, say) would be identically F, but nevertheless a theorem of axioms (a) to (e). P being identically F, $\neg P$ would be l.t. and so axioms (a)–(d) $\vdash \neg P$; *a fortiori* $\vdash \neg P$ in the enlarged system also, and our new axiom system lets us down by being inconsistent. The original calculus is complete then also in the sense that it admits of no consistent extension of its list of axioms. This impossibility of extending the range of the axioms suggests another question: can the list of axioms be shortened without reducing the class of theorems?

This is the same as asking: is any one of the axioms deducible from the rest, or are they all independent? It turns out that they *are* all independent, and a proof for one of them will be given below.

The remaining metamathematical problem is the decision problem: can one examine any sentence by means of some mechanical procedure and then, after a finite number of steps, correctly say "this is a provable sentence" or "this is not a provable sentence"? There are many decision procedures, amongst them the valuation and the normal form. If a sentence yields an entire value column of T's, then it is provable; if there is at least one F, then it is not. With the normal form, if every A_i contains a variable and its negation, then the sentence from which the normal form was derived is provable, and if there is at least one A_i lacking such a pair, the sentence is not provable.

By means of the valuation procedure, we showed that all sentences provable from axioms (a)–(d) by rules (i) and (ii) could be recognised by their having the value T irrespective of the values given to the constituent variables; if we can find another valuation which will characterise all sentences which are consequences of some three of the axioms, then show that the remaining axiom lacks this characteristic, we shall have established that this particular axiom is independent of the others. Four such valuations can be found, one for each axiom. Here is one for axiom (d): let the variables range over the four values 0, 1, 2, 3; let the functions $\neg$ and $\vee$ be defined as follows $(\neg 0) = 1$, $(\neg 1) = 0$, $(\neg 2) = 3$, $(\neg 3) = 0$;

$\alpha \vee 0 = 0 \vee \alpha = 2 \vee 3 = 3 \vee 2 = 0$, $\alpha \vee 1 = 1 \vee \alpha = \alpha \vee \alpha = \alpha$ $(\alpha = 0, 1, 2, 3)$. Axioms (a), (b), (c) turn out to be identically 0 under this valuation. For instance axiom (c), $\neg (p \vee q) \vee (q \vee p)$ has the form $(\neg \alpha) \vee \alpha$ which has a disjunct 0 whenever $\alpha = 0, 1, 3$, and is $3 \vee 2 = 0$ when $\alpha = 2$. The reader can test (a) and (b) for himself. Also, rule (i) applied to a sentence which is identically 0 yields another such, whilst if we infer B from A and $A \rightarrow B$ where $\alpha = 0$ and $(\neg \alpha) \vee \beta = 0$ (α, β being the values of A, B respectively), then, $(\neg 0)$ being 1, $(\neg \alpha) \vee \beta = 1 \vee \beta = \beta$, i.e. $\beta = 0$ also. Under this valuation, all consequences of axioms (a)–(c) are identically 0 then, but when p, q, r have the respective values 3, 1, 2, axiom (d) has the value 2, and so cannot follow from the others.

REFERENCES

[1] The sequence of derived rules and theorems is taken from Hilbert and Ackermann's "Principles of Mathematical Logic" (English version: Chelsea Pub. Co. 1949).

[2] The proof of the deduction theorem (originally due to Herbrand, 1928) is very like the one to be found in Kleene's "Introduction to Metamathematics" (Amsterdam, 1952) though Kleene provides himself

with an axiom schema (he uses axiom schemata rather than axioms) specially designed to carry out the induction step.

[3] Natural Inference methods were introduced by G. Gentzen in Mathematische Zeitschrift, Vol. XXXIX (1934).

[4] Truth tables as a decision procedure were first explicitly used by L. Wittgenstein ("Tractatus Logico-Philosophicus") and E. Post (Amer. Journal of Math., Vol. 43).

[5] The arithmetical procedures for establishing the independence of the axioms were found by P. Bernays.

[6] Completeness was first proved by J. Łukasiewicz.

The reader who wishes to read further, can do no better than turn to the work by Hilbert and Ackermann mentioned in [1]. The relevant sections of the work by Kleene are very thorough but do not spend much space on deductions from the axioms. "Mathematical Logic" by R. L. Goodstein was written with the same class of readers in mind as the present article, as was also the same author's article "The Decision Problem" (Gazette Vol. XLI, No: 335, Feb., 1957).

J. H.

Derby & District College of Technology.

1937. He (Thomas Hobbes 1588–1679) was forty yeares old before he looked on geometry; which happened accidentally. Being in a gentleman's library, Euclid's *Elements* lay open, and 'twas the 47 libri I. He read the proposition. 'By God,' sayd he, (He would now and then sweare, by way of emphasis), 'this is impossible!' So he reads the demonstration of it, which referred him back to such a proposition; which proposition he read. That referred him back to another, which he also read. *Et sic deinceps*, that at last he was demonstratively convinced of that trueth. This made him in love with geometry. (Aubrey: Brief Lives) [Per Mr. A. P. Rollett.]

1938. *Independence, or, is your axiom really necessary?*

Candidates offering History at **A** level must take Paper I, and two of the Papers II, III, IV, or V, with the following restrictions:

If IV is taken the third paper must be II.

If V is taken the third paper must be either II or III.

From the Regulations of the Oxford and Cambridge Schools Examination Board. [Per Mr. R. F. Wheeler.]

1939. "The Post Office loses 10 *per cent.* of its pens every month; that is 120 *per cent.* every year.—The Postmaster General." *The Observer*, 21 December, 1958. [Per Mr. G. N. Copley.]

1940. "In addition, it must be remembered that statistical correlations, however significant, do not necessarily imply a casual relationship."—J. L. Cloudesley-Thompson, *The New Scientist*, 1959; **5**, 1036. [Per Mr. G. N. Copley.]

OBITUARY

A. W. SIDDONS

"It is desirable that every schoolmaster should be intelligent, zealous, and stimulating." This is a quotation which Mr. Siddons gave from the sayings of his own schoolmaster, Mr. R. Levett. It is an understatement to say that the pupil followed this precept and went far beyond in the enthusiasm which he showed throughout his career. With the wide interests he had, he was primarily a teacher of mathematics, and he thought and talked about his work and sought by every possible means to maintain a high standard and inspire others with the same enthusiasm. He remained surprisingly free of set ideas which most teachers are subject to as they advance in age. In mathematical teaching he was an innovator himself and, while he would not abandon a well-tried idea without good reason, he did realise that there must be change and he was receptive of new ideas from any source, whether the workings of a grandson's mind or a remark from a young teacher at an Association meeting. The problems of introducing number and space to a child of 5 seemed to interest him as much as his Sixth-form work.

This notice must put on record the great work he has done for the Mathematical Association, and plain facts speak for themselves. Less than three years after he started teaching he was, in 1902, appointed the secretary of the first Teaching Committee of the Association, and in less than four months, their first report, on the teaching of Geometry, was published in the Gazette. He was also in the same year appointed to the Cambridge syndicate set up to consider mathematics in the pass examinations of the university. Thus he was able to spread the influence of the Association to wider fields, and the first steps in removing the Euclid order of theorems for all examinations followed quickly. It has been said of Mr. Siddons that the Teaching Committee in their first years must have owed much of their success to his combination of force of character with essential kindliness. He was not content to theorise, but his first two books, with Mr. C. Godfrey on Elementary Geometry and Elementary Algebra provided material for teachers and pupils to put into practice the new order and methods which he advocated.

His work with the Teaching Committee was so distinguished that in 1913 he was appointed a vice-president of the Association, an unusual honour for one as young as 37. In this way he held a position on the Council from that date till his death, and it was a rare event for him to miss a meeting. In the years following the first war he was prominent in all meetings of the Association and of

the London branch, and he was appointed president of the Association in 1935, the first schoolmaster to receive this honour. In his presidential address, published in vol. XX, no. 237, he gave a masterly account of the progress in teaching and the growth of the Association. He was finally elected an Honorary Member of the Association in 1958.

A. W. S. always considered that the Gazette should be primarily concerned with the teaching of elementary work, and was apt to say that successive issues were becoming too 'high-brow'. He often wrote and talked to the editor, with whom he sometimes disagreed, and put his case firmly, even bluntly, but always with such underlying kindliness that there was never any friction. Obeying the injunctions of the editor that if he wanted elementary articles he should write them, or get others to do so, he wrote frequently, and it is fitting that in this number his last contribution, written shortly before his death, should be printed.

As well as being a Wrangler in 1898 he also obtained a first class in part 1 of the Natural Science Tripos in 1899. While at Harrow he taught physics, as well as mathematics, for a long time, and aimed successfully at closer liaison between the mathematics and physics at school. He was concerned in his spare time in teaching scientists the beginnings of Calculus in the days before this was considered an essential part of their work.

To those who knew him in Harrow there are many more interests in Mr. Siddons' life which can here be referred to only briefly. While a master at the school he was twice in charge of the gymnasium. His greatest hobby was in gardening, and many gardens in Harrow, especially those of the School and the Hospital, have been reorganised by him, and show the breadth of his vision and constant work which he devoted to them. When he first retired from teaching he devoted much time to the Hospital, and became Chairman of the House Committee in 1943. He found time frequently to visit Dr. Bickley after he had lost his sight, reading to him to keep him up to date with modern developments. In the last few years, while his daughter was working on Braille at the institute for the Blind he became interested in the many problems of converting mathematical symbols to Braille. Above all, in his home with Mrs. Siddons, who died in 1954, he gave welcome to very many people, boys, masters, friends of all ages.

The steady growth of this Association which took its new name in 1897, only two years before he became a member bears testimony to the inspired work for teaching and wise guidance in Council which has always been his contribution.

K. S. SNELL

CORRESPONDENCE

To the Editor of the *Mathematical Gazette*

DEAR SIR,

In the Gazette No. 345 Miss Burslem writes of the difficulties children have in understanding Subtraction. This is not surprising. The fact is that those who have to teach subtraction do not know the meaning of the term. How then can they cause other than confusion in the minds of their pupils. The reason for this is partly because they are not mathematicians and partly because of the vicious circle of their having been taught incorrectly. Unfortunately, mathematicians who do not have to teach the subject have not bothered to give it the attention which is so urgently needed.

What then is the meaning of Subtraction? Simply this. We are given the sum of two quantities and one of them and we have to find the other. In other words, Subtraction is just an aspect of Addition. The phrase "Take away" is an artificiality which blinds us to the true meaning of things. It is obviously a deliberate invention of the devil to produce muddled thinking in the teacher and confusion in the minds of his pupils. The phrase "Take away" should be abolished and even the word Subtraction should not be used in the hearing of a pupil under the age of 15 by which time it is powerless to do any harm.

Millions of shop assistants and others who have to do subtractions turn the process into one of addition. To them the process seems perfectly natural and obvious. Many of them were dunces at school and got their subtractions wrong; now they always get them right.

This is how a subtraction should be arranged. Suppose that the sum of two numbers is 74 and one of them is 39; we have to find the other number. Put 74 where it ought to go, namely as the sum. Put the 39 as the first of the numbers to be added and leave a blank for the second number; this is our answer. The arrangement so far is,

39

—

74

—

Now add. (There should be some prior practice in adding downwards which is just as easy as adding upwards.) The process is 9 and *5* (put it down) are 14. Put down 4 (it is already down) and carry 1. 1 and 3 are 4 and *3* (put it down) are 7. The answer is *35*. If the answer is entered in red as is sometimes done in balancing an account this would add excitement and interest which are advantages I need not enlarge upon. However, this is not essential.

Incidentally, the answer is placed where it always appears when costing an account.

Of course, I am describing the method very briefly and I am not dealing with the gradual build-up which is needed in young children. One advantage of the method is that no new set of basic facts have to be mastered. At present, the pupil has to learn a set of basic facts for Subtraction as well as one for Addition. This unnecessary burden adds to his difficulties; no wonder he is so bothered.

Credit is due to Miss Burslem for her attempt to grapple with this problem. The searcher after truth has often to do more than one experiment and make a fresh start. I suggest that Miss Burslem starts again along the lines which I have indicated and—what is important—reports on her progress.

One last word. In the Association's Report on the Teaching of Arithmetic (page 18) the method recommended is that of Complementary Addition. And here are some of the names of the Committee which made this recommendation. F. C. Boon, C. T. Daltry, W. Hope Jones, A. Robson, A. W. Siddons, C. O. Tuckey, R. W. Wright, R. V. H. Roseveare, W. C. Fletcher—names to conjure with (they include four past Presidents of the Association). Is it not time we took their teaching to heart?

125 *Jersey Rd., Osterley.* Yours etc., S. INMAN

THE SCIENCE OF MECHANICS

To the Editor of the *Mathematical Gazette*

DEAR SIR,

In the October 1959 number of the Gazette there was an article with the above title. A sub-committee of the Teaching committee is now at work revising the report on the Teaching of Mechanics in Schools. We are at present working for this report along the lines suggested in the article.

In stage A we shall include topics of mechanics which can be introduced into the ordinary elementary course of mathematics in all schools, and which can serve to illustrate elementary methods and broaden the scope of examples.

In stage B we shall give the main course of mechanics and we hope that most schools may manage to start this, at any rate with better pupils, before examinations at "O" level are taken.

Stage C will suggest a course suitable for mathematical specialists, which can demonstrate the logical development of the subject from the assumptions laid down by Newton, and thus prepare students for the study of other fundamental assumptions, and the development which follows.

Yours etc., K. S. SNELL
(Chairman of the Mechanics Sub-committee)

COMPENDIUM OF EXAMPLES OF THE APPLICATIONS OF MATHEMATICS

To the Editor of the *Mathematical Gazette*

DEAR SIR,

Last year the Teaching committee appointed a new sub-committee to collect examples of the applications of Mathematics with the intention of publishing them later in a compendium. These examples are to be suitable for use in teaching boys and girls in secondary schools of all types. In the autumn of 1958 the sub-committee wrote to a large number of members of the association, who are teaching in technical colleges, to ask for their help and the response has been very good. Since then we have asked secretaries of branches of the Association to ask members of the branches to help us. We would however be very glad of the help of any member of the Association also and I will be very glad to send a copy of the letter, which we have sent out, to any member who writes to me. This letter states, at greater length than I can do here, the way in which we hope to compile the compendium.

Yours etc., H. V. LOWRY

198 *Epsom Rd., Guildford* (*Secretary of Sub-committee*)

To the Editor of the *Mathematical Gazette*

DEAR SIR,

According to your report in "The Mathematical Gazette", February 1959, page 39, three students at Reading School did a count of the frequencies of digits of π.

As for the 7's, it seems that their frequency 1001, is nearest the average. But let us examine this fact closely.

We shall count whole groups of digits.

There are 973 zeros (the average would be 1000),
there are 64 double zeros (the average would be 100),
there are 6 triple zeros (the average would be 10).

Now, the groups 2222 and 8888 have the exact average, 1; but the group 7777 appears four times! The group ·999999 ... appears once, in the beginning. Close to the end, the group 56001 appears twice.

For yet better investigation of π, we need more decimal places. There is also a French calculation of π in "Chiffres" I, 1, 1958.

Růžová 14, Yours etc., ADOLF KODYM
Prague 3,
Czechoslovakia

To the Editor of the *Mathematical Gazette*

DEAR SIR,

I am indebted to Mr. A. Kodym of Prague for pointing out that the expression for π as a continued fraction with unit numerators has been carried to 200 places by P. Pedersen (Nordisk Matematisk Tidskrift, Vol. 6, No. 2, 1958). In this paper it is shown that π appears to satisfy two conditions which a random number satisfies. Both the geometrical mean of the first n denominators and the nth root of the denominator of the nth convergent appear to converge (somewhat erratically) to the limiting values of the corresponding quantities for a random number.

I should like to correct the assertion in my previous letter (October 1959, p. 179) that the probability that, for a random number, any denominator is equal to r is $1/r(r+1)$. This is true only of the first denominator. Khintchine (Compositio Math. 3, pp. 276–285) has shown that the limit, as n tends to infinity, of the proportion of denominators which are equal to r is $\log[1 + 1/r(r+2)]/\log 2$. If we assume that 200 is a sufficiently large value of n for this to be approximately true, we obtain the following results.

Denominator	*Theoretical frequency*	*Actual frequency*
1	83	80
2	34	38
3	19	19
4	12	10
5	8	5
6	6	6
7	5	7
over 7	33	35

Finally, Khintchine (Compositio Math. 1, pp. 361–382) has shown that the arithmetic mean of the first n denominators of a random number tends asymptotically to $\log n/\log 2$. For $n = 200$ this value is 7·64. The corresponding value for π is 8·45. This agreement is quite good, since the arithmetic mean is dominated, to a much greater extent than the geometric mean, by the large denominators.

Yours etc., E. J. F. PRIMROSE

To the Editor of the *Mathematical Gazette*

DEAR SIR,

Your issue of October 1959 contains two letters on the numerical properties of π. The first letter, by Misses Curphey, Kelley, and Moffat, gives a frequency count of the first 10,000 digits of π, which I presume to be those computed by Mr. Felton and published in July 1957 in the *Proceedings of the Oxford Mathematical Conference*.

In April 1958, Mr. Felton informed me that this value of π was wrong after the 7480th decimal, due to a machine error.

The second letter, from Dr. Primrose, asks for an expression of π as a continued fraction with unit numerators and contains the sentence:—"For a random irrational number (assuming that "random" can be defined!) the probability that a denominator of the continued fraction is a given positive integer r is easily shown to be $1/r\,(r+1)$". This sentence may be compared with the following assertion:—"The shortest path between two points (assuming that "shortest" can be defined!) is easily shown to be a spiral.' In each case, the adjective in question can be defined in many different ways, and the truth of the statement will depend upon which definition is used. The appropriate definitions, to ensure the truth of the two statements, are somewhat pathological and unlikely to be relevant in normal contexts. Without essential loss of generality, we may suppose that the random number lies between 0 and 1 and therefore has the form

$$\Delta = \frac{1}{\delta_1{}^+}\,\frac{1}{\delta_2{}^+} \tag{1}$$

where $\delta_1, \delta_2, \ldots$ are positive integers. Amongst possible ways of defining the random character of Δ, *either* we may postulate $F(x) = \text{Prob}\,(\Delta \leq x)$ and deduce the joint distribution of $\delta_1\delta_2, \ldots$, *or* we may postulate the distribution of $\delta_1, \delta_2, \ldots$ and deduce the function $F(x)$.

If, for example, we adopt the first alternative together with the additional postulate that Δ is to be uniformly distributed between 0 and 1, we have $F(x) = x$. Then the probability that δ_1 and δ_2 will respectively equal r_1 and r_2 is

$$p(r_1, r_2) = F\left(\frac{1}{r_1{}^+}\,\frac{1}{r_2{}^+1}\right) - F\left(\frac{1}{r_1{}^+}\,\frac{1}{r_2}\right)$$

$$= \frac{1}{(r_1r_2 + 1)(r_1r_2 + r_1 + 1)} \tag{2}$$

Since the right-hand side of (2) does not factorize into the form $q_1(r_1)q_2(r_2)$, we see that δ_1 and δ_2 are not statistically independent. Further their marginal distributions are different. The probablity that δ_1 equals r_1 is

$$p_1(r_1) = \sum_{r_2=1}^{\infty} \frac{1}{(r_1r_2 + 1)(r_1r_2 + r_1 + 1)} = \frac{1}{r_1(r_1 + 1)}, \tag{3}$$

which is the form given by Dr. Primrose. But the probability that δ_2 equals r_2 is

$$p_2(r_2) = \sum_{r_1=1}^{\infty} \frac{1}{(r_1r_2 + 1)(r_1r_2 + r_1 + 1)} = \frac{\pi^2}{6r_2(r_2 + 1)} + O\left(\frac{1}{r_2{}^3}\right) \tag{4}$$

If, on the other hand, we adopt the second alternative together with the additional postulate that the δ's are statistically independent and are all distributed with the same probability law (3), suggested by Dr. Primrose, then it can be proved $F(x)$ has the form

$$F\left(\frac{1}{r_1}\frac{1}{r_2}\cdots\right) = \frac{1}{r_1}\left\{1 - \frac{1}{(r_1+1)r_2}\right\}\left\{1 - \frac{1}{(r_2+1)r_3}\right\}\left\{1 - \cdots \right. \quad (5)$$

The function given by (5) is a very pathological one, as the following properties show:—

(i) $F(x)$ is continuous and $F(0) = 0$ and $F(1) = 1$.

(ii) For almost all values of x, $F(x)$ is differentiable and its derivative equals 0.

(iii) $F(x)$ is rational if x is rational and also if x is a quadratic surd.

(iv) If x is rational, $F(x)$ has a left-hand and a right-hand derivative at x; and these two one-sided derivatives are equal if and only if the final denominator in the continued fraction form of x equals 3.

In (ii) the phrase (almost all x) means "all x which do not belong to a set of zero Lebesgue measure".

It can be shown, though I would not describe the proof as easy, that, for almost all values of Δ, the proportion of the first n denominators of Δ, which equal a prescribed integer r, tends to

$$\log_2\left\{\frac{(r+1)^2}{r(r+2)}\right\}$$

as $n \to \infty$. This and other results can be found in the following references:—

P. Lévy. *Théorie de l'Addition des Variables Aléatories* (pp. 311–313) Paris: Gauthier-Villars (1937).

P. Lévy. "Fractions continues aléatoires" *Rend. Circ. Mat. Palermo* (2) 1 (1952) 1–39.

C. Ryll-Nardzewski. "Ergodic theory of continued fractions". *Studia Math.* 12 (1951) 74–79.

Yours etc., J. M. HAMMERSLEY

To the Editor of the *Mathematical Gazette*

DEAR SIR,

Further to Dr. H. Martyn Cundy's letter and postscript about the "Stroud" system, not only was it given by Prof. Everett in 1879, but it was also "proposed" by M. Hospitalier at the International Congress of Electricians of 1891. Stroud was anticipated more than once!

Yours etc., T. W. HALL

To the Editor of the *Mathematical Gazette*

DEAR SIR,

The division sign ÷ and the proportionality sign : are in common use in schools, in school text-books and in school examination papers, while the solidus / is little used. On the other hand the solidus is in general use in the wider mathematical world whereas the division and proportionality signs are rare. I can see no good reason for this difference, and I suggest that the division and proportionality signs should be replaced by the solidus in school work. Can it seriously be asserted that the beginner would find it more difficult to comprehend

$$8\tfrac{1}{3}/3\tfrac{1}{3}, \qquad AB/AC = BD/DC$$

than

$$8\tfrac{1}{3} \div 3\tfrac{1}{3}, \qquad AB:AC::BD:DC?$$

One advantage of the solidus is that it enables the printer to print a literal fraction in the line of text without disturbing the spacing of the lines. Its use in school text-books would materially improve the appearance and reduce the cost.

Yours etc., F. C. POWELL

To the Editor of the *Mathematical Gazette*

DEAR SIR,

While the real part of the dinner served to members of the 1959 Annual Conference held in Southampton will doubtless live in the memories of the diners, I agree that the imaginary part is worthy of more permanent record and wider distribution. I should be sorry, however, to see thus permanently recorded an even more purely imaginary attribution of authorship, which I must—reluctantly!—disclaim. It was after my talk on Plaited Polyhedra that an elegantly constructed Scotch Terrier (woven, perhaps, rather than plaited, of strips of cardboard) was found outside the room of Dr. Maxwell, who promptly christened it Fidohedron; so I was suspected of that also—again incorrectly. So we are left with two mysteries; but the fun that the occurrences provoked was all part of the extremely happy atmosphere that pervaded the Conference throughout.

Yours etc., A. R. PARGETER

To the Editor of the *Mathematical Gazette*

DEAR SIR,

For some time I have been dissatisfied with the treatment of machines as found in many current textbooks. We frequently find definitions like these following which are quoted from a widely used

textbook of mechanics of advanced level standard:

"The ratio of the distance moved by the effort P to that moved by the weight W is called the Velocity Ratio."

"The efficiency of a machine is measured by the ratio

$$\frac{\text{Useful work done by the machine}}{\text{Work supplied to the machine}}.$$"

I admit that the phrase "in the same time", which surely ought to be added to both these definitions, is an almost self-evident addition, but it is hardly good teaching practice to furnish our pupils with incomplete definitions. In addition the question arises why the ratio of two distances is called Velocity Ratio.

I should be interested to know, whether any member can see serious objections to the following treatment which avoids the difficulties mentioned above and in which the quantities concerned are defined for any instant.

"The Velocity Ratio of a machine is the ratio of velocity of the effort to that of the load."

"The Efficiency of a machine is the ratio of the useful power obtained from the machine to that supplied to it."

Together with the relation

$$\text{Power} = \text{Force} \times \text{Velocity}$$

the relation between efficiency, velocity ratio and mechanical advantage is easily obtained, the mechanical advantage being defined in the usual way. It may be desirable to use speed instead of velocity in the first definition to avoid difficulties connected with the difference in direction of the velocities.

Yours etc., H. GEBERT

1941. "Compass and ruler are the only permissible tools in classical geometry. But geometry is the only language which enables man to understand the working of the divine mind. Therefore figures which cannot be constructed by compass and ruler—such as the septagon, the 11, 13 or 17-sided polygons—are somehow unclean, because they defy the intellect."—Arthur Koestler, *The Sleepwalkers*, p. 390. [Per Mr. G. N. Copley.]

1942. No doubt there was a certain justification for the classical mathematical problem about the logs and the elephant's task, in which the solver was permitted to "neglect the weight of the elephant," but no practical end was possible until the weight of the elephant was brought in. In our social and political discussions there are neglected elephants everywhere. We are all in a state of "flustered dogmatism" because of the unacknowledged presence of these exasperating animals.

H. G. Wells, The Anatomy of Frustration, 1936, p. 84. [Per Prof. E. H. Neville.]

2874. A note on perfect numbers

The following proof of the well known theorem that every even perfect number is of the form $2^n(2^{n+1} - 1)$, where $2^{n+1} - 1$ is prime, appears to be new.

Suppose that the even perfect number N is of the form

$$N = 2^n p_1^{\alpha_1} p_2^{\alpha_2} \dots p_s^{\alpha_s},$$

where n is a positive integer and p_i is prime. If $\sigma(N)$ is the sum of the divisors of N,

$$\sigma(N) = (2^{n+1} - 1) \prod_{i=1}^{s} (1 + p_i + \dots + p_i^{\alpha_i}) = 2N.$$

Hence $(2^{n+1} - 1)$ divides N and we may write

$$2^{n+1} - 1 = p_1^{\gamma_1} p_2^{\gamma_2} \dots p_s^{\gamma_s}.$$

It follows that

$$\frac{2N}{2^{n+1} - 1} = 2^{n+1} p_1^{\alpha_1 - \gamma_1} p_2^{\alpha_2 - \gamma_2} \dots p_s^{\alpha_s - \gamma_s} = \prod_{i=1}^{s} (1 + p_i + \dots + p_i^{\alpha_i}),$$

giving

$$(1 + p_1^{\gamma_1} p_2^{\gamma_2} \dots p_s^{\gamma_s}) p_1^{\alpha_1 - \gamma_1} p_2^{\alpha_2 - \gamma_2} p_s^{\alpha_s - \gamma_s} = \prod_{i=1}^{s} (1 + p_i + \dots + p_i^{\alpha_i}),$$

or

$$p_1^{\alpha_1 - \gamma_1} p_2^{\alpha_2 - \gamma_2} p_s^{\alpha_s - \gamma_s} + p_1^{\alpha_1} p_2^{\alpha_2} \dots p_s^{\alpha_s} = \prod_{i=1}^{s} (1 + p_i + \dots + p_i^{\alpha_i}).$$

Since $0 \leqq \gamma_i \leqq \alpha_i$ for all i, both the left-hand terms are included amongst those on the right. Hence the right-hand side can only contain two terms of this type, which is possible if and only if $\gamma = 0$ for all values of i except one, say $i = k$, for which $\alpha_k = 1$. It follows that $\gamma_i = 0$ for all values of $i \neq k$, and the above relation then reduces to

$$p_k^{1 - \gamma_k} + p_k = 1 + p_k.$$

It follows that $\gamma_k = 1$. Thus

$$N = 2^n p_k = 2^n(2^{n+1} - 1),$$

and $2^{n+1} - 1$ is prime.

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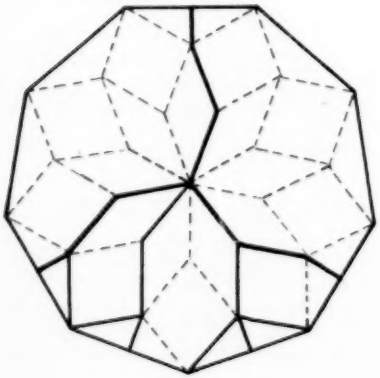
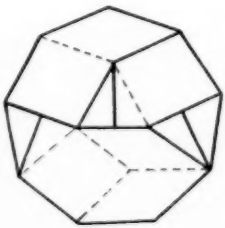
2875. On the dissection of a regular polygon into n equal and similar polygons

This problem clearly requires the construction of a length equal to the side of the polygon divided by $\sqrt{n}$. Now by drawing lines equal and parallel to the sides of the polygon a dissection into rhombi and triangles is effected, as for example in the cases given in Note 2788. In some cases the side of a triangle or the diagonal of one of the rhombi leads to the required length. Working empirically, C. D. Langford discovered that this could be done for the pentagon and decagon in the case $n = 5$ (Note 2628). The late E. Irving Freese of Los Angeles left on record an interesting general method for $n = 2$ when the polygon has $4k$ sides (k integral); in this case k rhombi will be squares (see the figures in Note 2788) and their half-diagonals give the required edges. By analogy with this, C. D. Langford correctly concluded that a polygon of $3k$ sides can be dissected to give k equilateral triangles; if their altitudes are drawn, the required ratio of $1 : \sqrt{3}$ can be obtained, and the problem solved for the case $n = 3$. His solution for the enneagon is here shown; the method is quite general.





The case of $n = 4$ is of course rational and can be effected by simple bisection of lengths; three general methods are known, one applicable to $3k$-gons, one to $4k$-gons, and one completely general. The dodecagon can thus be trisected by Langford's method, and quadrisected in three different ways.

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C. D. LANGFORD

2876. The Josephus problem

According to W. W. Rouse Ball in his book "Mathematical Recreations and Essays," the original problem dates back to Roman times and arose when forty-one Jews fleeing from the Romans took refuge in a cave. Fearing capture they decided to kill themselves. Two of the Jews did not agree with the proposal but were afraid to be open in their opposition. One of the two called Josephus suggested that they should arrange themselves in a circle and that counting around the circle in the same sense all the time, every third man should be killed until there was only one survivor who would then kill himself. By placing themselves in the 31st and 16th positions in the circle, Josephus and his companion saved their lives.

Problems of this type are easy to solve empirically. Considering a simpler case, it is easy to show that if a group of six people placed in a circle is reduced by removing every third person, the person who occupied the first position in the original circle is the survivor. If the group consists of seven people the survivor occupies the fourth position. This result however can be deduced from the previous one. The first person to be removed from the group of seven occupies the third position. For the six remaining people, counting starts at the fourth position. From the previous problem it is known that the survivor of a group of six occupies the position from which counting starts, in this case the fourth position. Thus, the addition of one person to the group advances the position of the survivor by three places. Similarly, for a group of eight people, if every third person is removed, the survivor is the seventh. In a group of nine people, the survivor would be the tenth person who is equivalent to the first person in the group.

This process can be generalized. Suppose that it is known that when every mth person is removed from a group of n, the survivor occupies the pth position in the original circle. The addition of x people to the circle places the survivor in the $(p + mx)$th position if this is less than or equal to $n + x$. If x is the lowest value for which $p + mx > n + x$, then the survivor occupies the position $(p + mx) - (n + x)$. Using this approach P. G. Tait was able to deduce many results. The result $n = 6$, $p = 1$, $m = 3$ can be used to solve the original problem. Thus with $x_1 = 3$, $n_1 = 9$, $p_1 = 1$; $x_2 = 5$ gives $n_2 = 14$, $p_2 = 2$; $x_3 = 7$ gives $n_3 = 21$, $p_3 = 2$; $x_4 = 10$ gives $n_4 = 31$, $p_4 = 1$. Finally with $x_5 = 10$, $n_5 = 41$ and $p_5 = 31$.

For convenience, the process used above to remove objects from a circle will be called reduction, and the number of steps, m, taken at each removal, the reduction constant.

A more difficult type of problem is illustrated by the following

example. The letters *PRQTSU* are arranged in that order around a circle. The problem is to decide whether the letters can be removed in alphabetical order by reduction, and if so, to find the value of the reduction constant. There are infinitely many solutions. Starting to count at U, $m = 60k + 32$ where k is a positive integer or zero. Counting from S in the reverse sense $m = 60k + 29$ where k is again a positive integer or zero. A simpler arrangement *PQRSUT* cannot be reduced for any value of m.

Problems similar to these may be solved easily with a different approach from that used by P. G. Tait. In what follows it will be assumed that removals are made while counting in the clockwise sense unless the contrary is stated.

Consider n objects $a_1, a_2, a_3, \ldots, a_n$ arranged in a circle. Regarding arrangements which can be obtained from a given arrangement by rotation as equivalent, there are $(n-1)!$ different arrangements possible. The arrangement in which the suffices are in ascending numerical order will be called the standard arrangement A_n. Any arrangement can then be written in the form σA_n where σ is a permutation of the symmetric group S_{n-1} of order $(n-1)!$ If e_r denotes the cyclic permutation $(r, r-1, \ldots, 1)$, then e_1 is the identity permutation and σ can be uniquely expressed in the form $e_2^{m_2} e_3^{m_3} e_4^{m_4} \ldots e_{n-1}^{m_{n-1}}$ where $0 \leqslant m_r < r$ and $2 \leqslant r \leqslant n-1$, r and m_r being integers.

The significance to be attached to the suffices is as follows. When σA_n is being reduced the first object to be removed is a_n, the next a_{n-1}, the rth a_{n-r+1} and the object left at the end is a_1. The permutation σ_{n-1}^m is such that $\sigma_{n-1}^m A_n$ can be reduced by the constant m in the order indicated by the suffices of the a's. In Tait's method numbers are attached to the objects according to their initial positions in the circle and the first object to be removed is the mth. In this method the objects are numbered according to the order in which they are removed from the circle the first to be removed being a_n.

It will now be shown that if $\sigma_{n-1}^m A_n$ can be reduced with the constant m then

$$\sigma_{n-1}^m = e_2^{m} e_3^{m} e_4^{m} \ldots e_{n-1}^{m} \qquad n \geqslant 3$$

The proof is by induction. When $n = 3$ there are just two possible arrangements

$$A_3 = e_2^{0} A_3 = \overset{a_1}{(}\ \overset{\frown}{}\ \overset{a_2}{)}_{a_3} \qquad \text{and} \qquad e_2^{1} A_3 = \overset{a_2}{(}\ \overset{\frown}{}\ \overset{a_1}{)}_{a_3}.$$

If a_2 is to be removed from A_3 after a_3, m must be even and $e_2^{0} = \sigma_2^{m}$, while in the case of $e_2 A_3$, m must be odd and $e_2^{1} = \sigma_2^{m}$. The result is thus true for $n = 3$.

Assuming the result to be true for $n = r$ we can say that $\sigma_{r-1}^{m} A_r$ can be reduced with the constant m. The removal of the object a_{r+1} from $\sigma_r{}^m A_{r+1}$ must leave $\sigma_{r-1}^{m} A_r$.

Let

$$\sigma_{r-1}^{m} = \begin{pmatrix} 1 & 2 & 3 & \dots & r-1 \\ b_1 & b_2 & b_3 & \dots & b_{r-1} \end{pmatrix}$$

then the first column of the table below shows the order of the suffices of the a's in A_r and the second column shows the order of the suffices in $\sigma_{r-1}^{m} A_r$

A_r	$\sigma_{r-1}^{m} A_r$	$\sigma_r{}^m A_{r+1}$	A_{r+1}
1	b_1	b_1	$m+1$
2	b_2	b_2	$m+2$
3	b_3	b_3	$m+3$
⋮	⋮	⋮	⋮
$r-m+1$	b_{r-m-1}	b_{r-m-1}	$r-1$
$r-m$	b_{r-m}	b_{r-m}	r
$r-m+1$	b_{r-m+1}	$r+1$	$r+1$
$r-m+2$	b_{r-m+2}	b_{r-m+1}	1
⋮	⋮	⋮	⋮
$r-1$	b_{r-1}	b_{r-2}	$m-2$
r	r	b_{r-1}	$m-1$
		r	m

Since a_r is to be removed after a_{r+1} from $\sigma_r{}^m A_{r+1}$ it follows that a_{r+1} must be between a_{b_r-m} and a_{b_r-m+1} as shown in the third column. The last column lists the suffices of A_{r+1} so that $\sigma_r{}^m$, the permutation of the first r integers may be found. From the last two columns

$$\sigma_r{}^m = \begin{pmatrix} 1 & 2 & \dots & m-1 & m & m+1 & \dots & r-1 & r \\ b_{r-m+1} & b_{r-m+2} & \dots & b_{r-1} & r & b_1 & \dots & b_{r-m-1} & b_{r-m} \end{pmatrix}$$

$$= \sigma_{r-1}^{m}\begin{pmatrix} b_1 & b_2 \dots & b_{r-1} \\ 1 & 2 \dots & r-1 \end{pmatrix}\begin{pmatrix} 1 & 2 & \dots & r \\ b_{r-m+1} & b_{r-m+2} & \dots & b_{r-m} \end{pmatrix}$$

$$= \sigma_{r-1}^{m}\begin{pmatrix} 1 & \dots & m-1 & m & m+1 & \dots & r \\ r-m+1 & \dots & r-1 & r & 1 & \dots & r-m \end{pmatrix}$$

$$= \sigma_{r-1}^{m} e_r^{\,m}(1, \dots, r)^m$$

$$\times \begin{pmatrix} 1 & \dots & m-1 & m & m+1 & \dots & r \\ r-m+1 & \dots & r-1 & r & 1 & \dots & r-m \end{pmatrix}$$

$$= \sigma_{r-1}^{m} e_r^{\,m}$$

Hence if the result is true for $n = r$ it is also true for $n = r + 1$. Since it is true for $n = 3$ it must be so for all positive integers greater than 2.

The converse result is that the arrangement $e_2^{m_2} e_3^{m_3} \dots e_{n-1}^{m_{n-1}} A_n$ can be reduced with the constant m if a number m can be found which satisfies the equations $m \equiv m_r \pmod{r}$ $2 \leqslant r \leqslant n-1$. This can also be proved. If m cannot be found due to inconsistencies in the residues, then the arrangement cannot be reduced for any constant m. For this reason only twelve of the twenty four possible permutations of A_5 can be reduced.

The fact that the object a_r is removed by taking m steps after removing a_{r+1} is indicated in the permutation σ_{n-1}^{m} by the term $e_r^{\,m}$.

If N is the L.C.M. of the first $n-1$ positive integers then

$$\sigma_{n-1}^{m} = \sigma_{n-1}^{N+m} = \sigma_{n-1}^{2N+m} = \dots = \sigma_{n-1}^{kN+m}$$

where k is a positive integer, indicating that if m is a reduction constant, then $kN + m$ is also.

So far only clockwise removals have been considered. The question now arises, given an arrangement $\sigma_{n-1}^{m} A_n$, can it be reduced in the anticlockwise sense with a constant m', and if so, what is the relationship between m, n and m'? It will be shown that an arrangement that can be reduced in the clockwise sense can always be reduced in the anticlockwise sense, and that $m' = N - m + 1$ where N is the L.C.M. of the first $n-1$ positive integers.

Let B_n represent the standard arrangement in the anticlockwise sense. This arrangement can be reduced in the clockwise sense when $m = 1$ from which $B_n = \sigma_{n-1}^{1} A_n$, $A_n = \sigma_{n-1}^{1} B_n$ and $\sigma_{n-1}^{1}\sigma_{n-1}^{1} = e_1$. If we assume that the arrangement $\sigma_{n-1}^{m} A_n$ can be reduced in the anticlockwise sense with the constant m', then it must be possible to write $\sigma_{n-1}^{m} A_n = \sigma_{n-1}^{m'} B_n$ or $\sigma_{n-1}^{m}\sigma_{n-1}^{1} = \sigma_{n-1}^{m'}$. When the reduction of $\sigma_{n-1}^{m} A_n$ has reached the stage represented by $\sigma_r^{\,m} A_{r+1}$, there are $r+1$ objects left. In order that a_r may be removed after a_{r+1}, there must be m steps from a_r to a_{r+1} in

the clockwise sense. Hence in the anticlockwise sense there must be $r + 1 - m$ steps giving rise to the term $e_r^{r+m-1} = e_r^{1-m}$ in the permutation $\sigma_{n-1}^{m'}$. It follows that

$$\begin{aligned}\sigma_{n-1}^{m'} &= e_2^{1-m} e_3^{1-m} \dots e_{n-1}^{1-m} \\ &= e_2^{N-m+1} e_3^{N-m+1} \dots e_{n-1}^{N-m+1} \\ &= \sigma_{n-1}^{N-m+1}\end{aligned}$$

where N is the L.C.M. of the first $n - 1$ positive integers. This means that if an arrangement can be reduced in one sense then it can always be reduced in the opposite sense the constants being related by the equation $m + m' = N + 1$. Algebraically this is represented by the equations

$$\sigma_{n-1}^{m} A_n = \sigma_{n-1}^{m} \sigma_{n-1}^{1} B_n = \sigma_{n-1}^{N-m+1} B_n.$$

It is impossible to reduce an arrangement with the same constant in both senses since this would mean that $2m = N + 1$. However N must be even and m and N are integral so that the equation cannot be satisfied. One of the constants m and m' must be odd and the other even.

It is now possible to deduce the value of m by which r objects may be removed from consecutive positions. The arrangement is $a_n, a_{n-1}, \dots, a_{n-r+2}, a_{n-r+1}, \dots$ the last $n - r$ objects being some arrangement of $a_1, a_2, \dots, a_{n-r}$. The part of σ_{n-1}^{m} concerned with the removal of the first r objects is $e_{n-r+1}^{1} e_{n-r+2}^{1} \dots e_{n-1}^{1}$. Thus $m = 1 + Ns$ where N is the L.C.M. of $n - r + 1, n - r + 2, \dots, n - 1$ and s is a positive integer or zero. In the reverse sense $m' = Ns$.

There is another category of problems which can be treated simply by the above method. In these problems the objects are divided into two or more groups and the groups have to be removed in turn. A simple example is to find the methods of reducing the arrangement $PPQQPQ$ so that those objects occupying the positions indicated by the P's are removed first. If a_6 occupies the first position $m = 11 \pmod{20}$ or $14 \pmod{20}$, if a_6 occupies the second position $m = 15 \pmod{20}$ or $18 \pmod{20}$ and in the last case $m = 8 \pmod{20}$ or $17 \pmod{20}$. In each case $m' = 21 - m$.

An example involving three groups is to remove the objects occupying the positions P first, then those occupying the positions Q, from the arrangement $PQPQRR$. If a_6 lies in the first position $m = 57 \pmod{60}$ or $52 \pmod{60}$ while if a_6 occupies the third position $m = 49 \pmod{60}$ or $54 \pmod{60}$. In both cases $m' = 61 - m$.

Finally a problem due to H. E. Dudeney which is quoted from Rouse Ball's book. "Let five Christians and five Turks be arranged

around a circle thus *TCTCCTCTCT*. Suppose that, if beginning at the ath man, every hth man is selected, all the Turks will be picked out, but if beginning at the bth man every kth man is selected all the Christians will be picked out. The problem is to find a, b, h, and k. A solution is $a = 1$, $h = 11$, $b = 9$, $k = 29$." A complete solution of the problem is given in the table below. The reduction constants can be increased by integral multiples of 504, the L.C.M. of 6, 7, 8, and 9.

h	k	a	b			Clockwise solns.	h	k	a	b		
k'	h'			a'	b'	Anti-clockwise solns.	k'	h'			a'	b'
237	268	4	8	4	10		368	137	6	6	2	2
215	290	6	6	2	2		66	439	8	4	10	4
476	29	3	9	5	9		11	494	1	1	7	7
395	110	4	8	4	10		335	170	7	5	1	3
38	467	1	1	7	7		158	347	4	8	4	10
410	95	7	5	1	3		338	167	4	8	4	10
86	419	1	1	7	7		412	93	10	2	8	6
443	62	4	8	4	10		250	255	2	10	6	8
218	287	9	3	9	5		261	244	1	1	7	7
44	461	10	2	8	6		423	82	9	3	9	5

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2877. A trigonometrical inequality

Two well-known inequalities are

$$\frac{2}{\pi^2} \leqslant \frac{1-\cos\theta}{\theta^2} \leqslant \frac{1}{2}, \qquad \frac{1}{\pi^2} \leqslant \frac{\theta - \sin\theta}{\theta^3} \leqslant \frac{1}{6}, \tag{1}$$

both of which are valid for $0 \leqslant \theta \leqslant \pi$. Both are special cases of a general inequality given below [see (8) and (9)].

Let $F_n(\theta)$ be defined as, for $n \geqslant 1$ and $0 < \theta \leqslant \pi$,

$$F_n(\theta) = \frac{1}{(n-1)!}\int_0^\theta (\theta - \phi)^{n-1} \sin\phi \, d\phi \, . \left\{\frac{\theta^{n+1}}{(n+1)!}\right\}^{-1}; \tag{2}$$

the range of definition may be extended to include $\theta = 0$ by defining $F_n(0)$ as the limit of $F_n(\theta)$ as $\theta \to 0$. An application of l'Hospital's rule then shows that $F_n(0) = 1$ $(n \geqslant 1)$.

We can now state the following result (a prime denotes differentiation with respect to θ):

$$F_n'(\theta) < 0 \text{ for } 0 < \theta \leqslant \pi \quad F_n'(0) = 0 \ (n \geqslant 1) \tag{3}$$

To see this we have to consider the cases $n = 1$ and $n > 1$ separately. For $n = 1$

$$F_1(\theta) = 2\theta^{-2} \int_0^\theta \sin \phi \, d\phi$$

and thus, for $\theta \neq 0$,

$$\begin{aligned} \tfrac{1}{2}\theta^4 F_1'(\theta) &= \theta^2 \sin \theta - 2\theta \int_0^\theta \sin \phi \, d\phi \\ &= \theta^2 \sin \theta - \theta[\phi \sin \phi]_0^\theta + \theta \int_0^\theta \phi \cos \phi \, d\phi - \theta \int_0^\theta \sin \phi \, d\phi \\ &= -\theta \int_0^\theta \{\sin \phi - \phi \cos \phi\} \, d\phi \end{aligned} \tag{4}$$

When $n > 1$ we have, on differentiating (2) directly, for $\theta \neq 0$

$$\begin{aligned} \left\{\frac{\theta^{n+1}}{(n+1)!}\right\}^2 F_n'(\theta) &= \frac{\theta^{n+1}}{(n+1)!} \frac{1}{(n-2)!} \int_0^\theta (\theta - \phi)^{n-2} \sin \phi \, d\phi \\ &\quad - \frac{\theta^n}{n!} \frac{1}{(n-1)!} \int_0^\theta (\theta - \phi)^{n-1} \sin \phi \, d\phi \\ &= \frac{\theta^n}{(n+1)!(n-1)!} \int_0^\theta (\theta - \phi)^{n-2} \{n\phi \\ &\quad + \phi - 2\theta\} \sin \phi \, d\phi \end{aligned} \tag{5}$$

Now

$$\begin{aligned} &\int_0^\theta (\theta - \phi)^{n-2} \{n\phi + \phi - 2\theta\} \sin \phi \, d\phi \\ &\quad = (n-1) \int_0^\theta (\theta - \phi)^{n-2} \phi \sin \phi \, d\phi \\ &\qquad - 2 \int_0^\theta (\theta - \phi)^{n-1} \sin \phi \, d\phi = [-(\theta - \phi)^{n-1} \phi \sin \phi]_0^\theta \\ &\qquad + \int_0^\theta (\theta - \phi)^{n-1} \{\sin \phi + \phi \cos \phi\} \, d\phi \\ &\qquad - 2 \int_0^\theta (\theta - \phi)^{n-1} \sin \phi \, d\phi \\ &\quad = -\int_0^\theta (\theta - \phi)^{n-1} \{\sin \phi - \phi \cos \phi\} \, d\phi \end{aligned} \tag{6}$$

From (4), (5), and (6) it now follows that, for $n \geqslant 1$ and $0 < \theta \leqslant \pi$,

$$F_n'(\theta) = -\frac{(n+1)!}{(n-1)!}\theta^{-(n+2)}\int_0^\theta (\theta-\phi)^{n-1}\{\sin\phi - \phi\cos\phi\}\,d\phi \quad (7)$$

Clearly $(\theta-\phi)^{n-1} \geqslant 0$ for $0 \leqslant \phi \leqslant \theta \leqslant \pi$ and it is known that $\sin\phi - \phi\cos\phi \geqslant 0$ for $0 \leqslant \phi \leqslant \pi$ (with equality only if $\phi = 0$); from this it follows that the right-hand side of (7) is non-positive for $0 < \theta \leqslant \pi$ and, in fact, strictly negative unless $\theta = 0$. This, together with a further application of l'Hospital's rule as $\theta \to 0$, establishes the result stated in (3).

An application of (3) now gives the following trigonometrical inequality

$$F_n(\pi) \leqslant F_n(\theta) \leqslant F_n(0) = 1 \quad (0 \leqslant \theta \leqslant \pi \quad n \geqslant 1) \qquad (8)$$

with equality at the two limits only if $\theta = \pi$ and $\theta = 0$ respectively. The inequalities (1) follow as special cases of (8) for $n = 1$ and 2 respectively.

The inequality (8) can be stated more explicitly if we use the following expressions for $F_n(\theta)$:

$$F_n(\theta) = (-1)^s\left[\cos\theta - \sum_{t=0}^{s-1}(-1)^t\frac{\theta^{2t}}{(2t)!}\right]\left\{\frac{\theta^{n+1}}{(n+1)!}\right\}^{-1} \text{ if } \begin{matrix} n = 2s-1 \\ s \geqslant 1 \end{matrix} \qquad (9)$$

$$= (-1)^s\left[\sin\theta - \sum_{t=0}^{s-1}(-1)^t\frac{\theta^{2t+1}}{(2t+1)!}\right]\left\{\frac{\theta^{n+1}}{(n+1)!}\right\}^{-1} \text{ if } \begin{matrix} n = 2s \\ s \geqslant 1 \end{matrix}$$

Finally if we use the result (3) directly for the cases $n = 1, 2, 3,$ and 4 we obtain the following inequalities, respectively, all valid for $0 \leqslant \theta \leqslant \pi$,

$$\frac{\sin\theta}{\theta} \leqslant \frac{2(1-\cos\theta)}{\theta^2}, \qquad \frac{\sin\theta}{\theta} \leqslant \frac{2+\cos\theta}{3},$$

$$\frac{\sin\theta}{\theta} \geqslant \frac{4-4\cos\theta-\theta^2}{\theta^2}, \qquad \frac{\sin\theta}{\theta} \geqslant \frac{\cos\theta+4-\theta^2/3}{5},$$

with equality only when $\theta = 0$.

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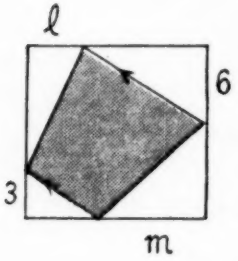
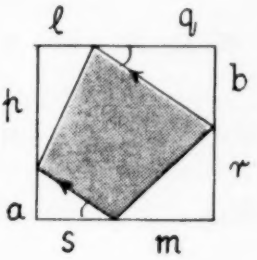
2878 Ship aground—A land-lubber's adventure

The following exercise, which I composed in a sadistic moment, always seemed to me a piece of square-rigging! It was, in fact, a "rider" of turbulent seas and not a few young sailors sickened because of it! Nemesis, however, overtook me. I ran my barque aground and had, laboriously, to refloat it!

The sail, as normally set:—

Trapezium, inscribed in square. 3, l, and 6 are in H.P. l, 6, and m are in H.P. Find (i) l (ii) m (iii) side of square. Prove (iv) Area of Trap. = 222 sq units.





The cause of disaster:—

Thoughtless interchange, on one occasion, of 3 and 6. "What, then," said the master, "Is the sail-spread?"

The complaint lodged by the ship-wrecked crew:—

Side of square < 6. Danger of "external section of side(s)," had not occurred to the said master.

The Inquiry:—

If all the line-segments are positive numbers, to find within what limits the ratio, b/a, lies.

We have,

$$l = \frac{2ab}{a+b}$$

Hence

$$m = \frac{2ab}{3a-b}$$

If x be the length of the side,

$$\frac{a}{x-m} = \frac{b}{x-l} \quad \text{leads to} \quad x = \frac{2ab(3a+b)}{(a+b)(3a-b)}$$

and hence

$$s = \frac{4a^2b}{(a+b)(3a-b)} \qquad q = \frac{4ab^2}{(a+b)(3a-b)}$$

$$r = \frac{b(3a^2+b^2)}{(a+b)(3a-b)} \qquad p = \frac{a(3b^2+4ab-3a^2)}{(a+b)(3a-b)}$$

It will be seen that x, q, r, and s are positive if $b/a < 3$.

Now

$$p = \frac{a}{(a+b)(3a-b)}(3b^2 + 4ab - 3a^2)$$

$$\frac{a}{(a+b)(3a-b)} \text{ is positive for } \frac{b}{a} < 3$$

Consider $3b^2 + 4ab - 3a^2$. This is positive if

$$3b^2 + 4ab > 3a^2$$

if $$3\left(b + \frac{2}{3}a\right)^2 > \frac{13}{3}a^2$$

if $$\left(b + \frac{2}{3}a\right) > \frac{\sqrt{13}}{3}a$$

if $$\frac{b}{a} > \frac{\sqrt{13}-2}{3}$$

Hence $\frac{b}{a}$ must lie between $\frac{\sqrt{13}-2}{3}$ and 3.

In the original example, $\frac{b}{a} = \frac{6}{3} = 2$, which is within the limits, but $\frac{b}{a} = \frac{3}{6} = \frac{1}{2}$ is outwith the limits.

The Findings:—

Let this be a salutary lesson to ship "doctors" to taste first the medicine they prescribe; and to masters, to set their courses within the mathematically prescribed limits.

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2879. Linear differential equations

The demonstration of the form of the second solution of the linear differential equation of the second order, when the auxiliary equation has equal roots, given in Note 2823, (343, p. 45) is logically unexceptionable, but suffers, I submit, from two serious disadvantages. In the first place, will not students—both intelligent and otherwise—ask *why* the constant A is rewritten in the complicated form which leads to the final result? Secondly—and this is a more serious and general criticism of much of the writing on differential equations—a differential equation is, in itself, not a *complete* problem. To make a solution precise or unique, additional information, by way of initial or boundary conditions, is required.

In the case of the differential equation symbolically expressed as

$$(D - \lambda)(D - \mu)y = 0,$$

subjoin the conditions that, when $x = 0$, $y = Y$, and $dy/dx = V$. These conditions entail that

$$A + B = Y,$$

$$\lambda A + \mu B = V,$$

or,

$$A = (\mu Y - V)/(\mu - \lambda), \quad B = (\lambda Y - V)/(\lambda - \mu).$$

Inserting these values of A and B into the solution

$$y = Ae^{\lambda x} + Be^{\mu x},$$

and then taking the limit as μ tends to equality with λ, we derive the solution

$$y = \{Y + (V - \lambda Y)x\}e^{\lambda x}$$

W. G. BICKLEY

2880. On Note 2835.

The theorem proved by D. S. Mitrinovitch can be established more concisely as a direct consequence of Fermat's theorem, as follows:—

Let n be a positive integer and let $p, q, r, \ldots$ be primes. Then the definition $E(n)$ becomes

$$E(n) = n[1 - (1/p + 1/q + 1/r + \ldots)] + n^p/p + n^q/q + n^r/r + \ldots$$

$$= n + [(n^p - n)/p + (n^q -)/q + (n^r - n)/r + \ldots].$$

Now by Fermat's theorem all the terms in the square brackets are integers. Hence $E(n)$ is an integer.

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2881. On Note 2848.

I have obtained a number of solutions of the system of equations at the end of Note 2848. The following is the one with the smallest value of a.

$$\tfrac{1}{10} = \tfrac{1}{11} + \tfrac{1}{110} = \tfrac{1}{12} + \tfrac{1}{16} = \tfrac{1}{14} + \tfrac{1}{35}$$
$$= \tfrac{1}{13} + \tfrac{1}{65} + \tfrac{1}{130} = \tfrac{1}{15} + \tfrac{1}{31} + \tfrac{1}{930}$$
$$= \tfrac{1}{16} + \tfrac{1}{40} + \tfrac{1}{80} = \tfrac{1}{17} + \tfrac{1}{34} + \tfrac{1}{85}$$
$$= \tfrac{1}{18} + \tfrac{1}{30} + \tfrac{1}{90} = \tfrac{1}{20} + \tfrac{1}{21} + \tfrac{1}{420}$$

Portland Modern School, Birmingham 17 D. C. CROSS

49. The circular points

In a scholarship paper set a few years ago by a group of Cambridge colleges the following question occurred:—

"Consider the two propositions:—

(i) The tangents at two points I, J of a conic meet at O, and a given line through O meets the conic in A, B; then the lines joining any point P of the conic to A, B meet the line IJ in points separating I, J harmonically;

(ii) AB is a given diameter of a circle; then the lines joining any point P of the circle to A, B are perpendicular.

Sketch the theory of the circular points at infinity by which you would justify the use of proposition (i) to prove (ii), *or* (if you prefer it) the use of proposition (ii) to prove (i)."

There now follows a suggested sketch of the theory, too full for the time limits of an examination answer though obviously still a sketch and to be amplified in class, but offered here to provoke comment and criticism. On the one hand it may be that it represents a level of understanding which a schoolboy should reach and that a discussion in school of this sort of problem on these lines would help to eliminate much of the muddled thought which often follows the words "Projecting A, B into the circular points at infinity...": on the other hand some people may think that the matter has been oversimplified in this sketch and that boys brought up on this fare would be underdeveloped; they would expect something more thorough to be taught.

In what follows we shall, for brevity, call a complex number "real" if its imaginary part is zero.

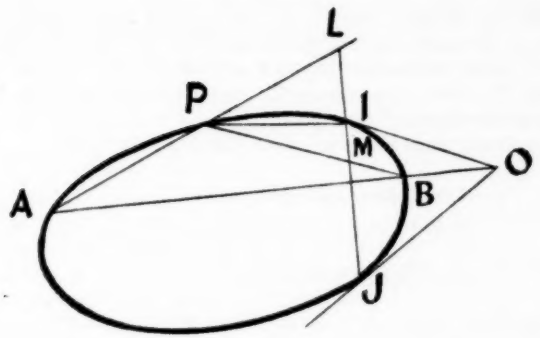
1. Proposition (i) is a theorem about certain lengths in a configuration $\mathscr{E}$ of the Euclidean Plane. We call the theorem T_1 and the Geometry E.

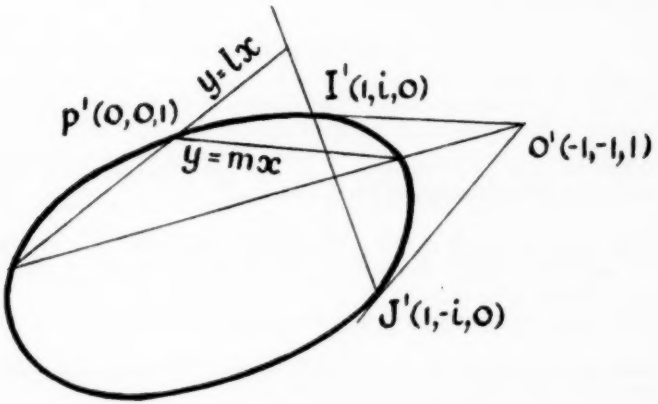
Let PA, PB meet IJ in L, M respectively, then T_1 states that

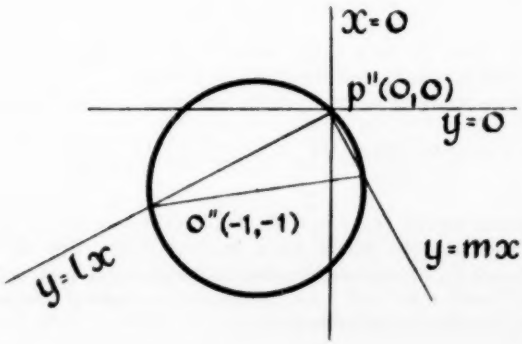
$$P\{IJ;\ LM\} = -1.$$

2. We take Cartesian axes and identify points by their coordinates. This gives us Real Cartesian Geometry, C. The "configuration" $\mathscr{C}$ is the set of coordinates identifying $\mathscr{E}$. The theorem T_2 concerns the cross-ratio of 4 numbers. Its truth follows from that of T_1, for the numbers represent the lengths in $\mathscr{E}$.

3. We now *define* a point as a triple of numbers (x, y, z), complex and not all zero. If $k \neq 0$ then (kx, ky, kz) and (x, y, z) are the same point. This gives us the geometry of the Complex Projective Plane (C.P.P.).


L
P
I
M
O
A
B
J


y=lx
I'(1,i,0)
p'(0,0,1)
y=mx
O'(-1,-1,1)
J'(1,-i,0)


x=0
p''(0,0)
y=0
O''(-1,-1)
y=lx
y=mx

There is a subset of points, in which $z = 1$ and x, y are "real", having a Geometry indistinguishable from C. The configuration (in this subset) corresponding to $\mathscr{C}$ we call $\mathscr{P}$; it is described by a theorem T_3 about the cross-ratio of 4 numbers. T_2 and T_3 are indistinguishable algebraically.

We make a transformation of the C.P.P. of the form:—

$$\left.\begin{aligned} x' &= \lambda_{11}x + \lambda_{12}y + \lambda_{13}z \\ y' &= \lambda_{21}x + \lambda_{22}y + \lambda_{23}z \\ z' &= \lambda_{31}x + \lambda_{32}y + \lambda_{33}z \end{aligned}\right\} \text{ where } |\lambda_{ij}| \neq 0.$$

The notation to be used is that this transforms $\mathrm{Q}(x, y, z)$ into $\mathrm{Q}'(x', y', z')$ for all Q, and transforms the configuration $\mathscr{P}$ into $\mathscr{P}'$. We can choose the λ_{ij} so that

$$P' \text{ is } (0, 0, 1)$$

$$I' \text{ is } (1, i, 0)$$

$$J' \text{ is } (1, -i, 0).$$

Let O' be $(-g, -f, 1)$ since O', I', J' are not collinear; I', J' are called the **circular points at infinity** because:

(a) they lie on all conics in the C.P.P. which give rise by the present process to circles in the Euclidean Plane, to which we are returning.
(b) they are not members of the subset in which $z = 1$.

The conic of $\mathscr{E}$ gives rise to a conic Σ' in the C.P.P. passing through P', I', J'; also the polar of O' is $I'J'$; so Σ' is

$$x^2 + y^2 + 2fyz + 2gzx = 0.$$

Let $P'L'$ meet Σ' in a point with "real" coordinates; say $P'L'$ $y = lx$, with l "real"; and let $P'M'$ be $y = mx$.

The cross-ratios of related pencils are equal and so T_3 yields:—

$$\frac{(l - i)(m + i)}{(l + i)(m - i)} = -1 \quad \text{or} \quad lm + 1 = 0.$$

T_3' is the result: $lm + 1 = 0$. It is a theorem about the lines $y = lx$ and $y = mx$ which meet at $P'(0, 0, 1)$. For all "real" l (and so m) T_3' is expressible entirely in terms of points (x, y, z) with $z = 1$ and x, y "real"; i.e. in terms of a subset of points whose geometry is indistinguishable from C.

4. The truth of T_3' ensures that there is a theorem T_4 about a configuration $\mathscr{C}'$ whose geometry is C.

$\mathscr{C}'$ consists of: the curve $x^2 + y^2 + 2gx + 2fy = 0$
a point $P''(0, 0)$ on the curve
the lines $y = lx$, $y = mx$ passing one each through the 2 points where a line through $O''(-g, -f)$ cuts the curve.

T_4 is indistinguishable algebraically from T_3'; it is that $lm + 1 = 0$ i.e. that the lines of the Euclidean plane represented in C by $y = lx$ $y = mx$ are at right angles.

5. T_4 is the description in C of the proposition (ii) of E. The truth of T_4 ensures the truth of proposition (ii).

J. H. DURRAN

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50. The remainder theorem again

The doubts that still exist in some minds about the validity of the usual school proof of the Remainder Theorem and which were referred to in Class Room Note 23 in Vol. XLII, No. 341 have prompted me to return to this topic.

Let P_r be a polynomial of degree r in x; $r = 0, 1, 2, \ldots$. So P_4 will denote an expression of type $cx^4 + dx^3 + ex^2 + fx + g$, P_1 one of type $ax + b$ and P_0 a constant—where $a, b, c, \ldots$ are independent of x. In the so-called division process below, we suppose only $a, c \neq 0$.

$P_1)\ P_4 \qquad\qquad (q_1 + q_2 + q_3 + q_4$

$$\frac{q_1P_1}{P_4 - q_1P_1}$$

$$\frac{q_2P_1}{P_4 - (q_1 + q_2)P_1}$$

$$\frac{q_3P_1}{P_4 - (q_1 + q_2 + q_3)P_1}$$

$$\frac{q_4P_1}{P_4 - (q_1 + q_2 + q_3 + q_4)P_1 = P_0,}$$ the remainder free from x.

Hence $P_4 \equiv (q_1 + q_2 + q_3 + q_4)P_1 + P_0$.

To find q_1, cx^4 is divided by ax; but the equality $\frac{cx^4}{ax} = \frac{c}{a}x^3$ or q_1 is not used: it would forbid x to have the value 0. Instead it is the multiplication version, $cx^4 = \frac{c}{a}x^3 \,.\, ax$ which is employed in the

process: this holds for *all* values of x. Also, the first subtraction holds for *all* values of x, yielding as remainder $P_4 - q_1P_1$, a polynomial of degree 3 at most. The rest of the division, until the remainder P_0 free from x is reached, is but a repetition of the processes already described. What has been established is that corresponding to given polynomials in x P_1, P_4 a polynomial $q_1 + q_2 + q_3 + q_4$ or P_3, say, and a constant P_0 *exist* such that

$$P_4 \equiv P_1 \,.\, P_3 + P_0 \tag{1}$$

where x is totally unrestricted. This is what the division process primarily shows beyond doubt. The common misapprehension surrounding the usual proof of the Remainder Theorem centres on the illusion that the identity just written derives from the almost equivalent one

$$\frac{P_4}{P_1} \equiv P_3 + \frac{P_0}{P_1} \tag{2}$$

Instead, this is derived from (1), with the limitation that x must not make P_1 vanish!

Knowing that an identity such as (1) *exists*, we can without division, by the method of undetermined coefficients find the forms of P_3, P_0 completely by substituting $(4 + 1)$ values of x in the assumed form. Usually, however, only P_0 is required, and one substitution suffices to fix this (in ignorance of P_3): it is the value that makes P_1 vanish. Hence P_0, the remainder, is the result of substituting in the dividend P_4 that value of $x\left(\text{namely} -\frac{b}{a}\right)$ which makes P_1 vanish. Similar remarks apply when the dividend is P_r instead of P_4.

The "division" process likewise leads to the result that corresponding to given polynomials P_s, P_r $(s \leqq r)$, other polynomials P_{r-s}, P_{s-1} exist such that

$$P_r \equiv P_sP_{r-s} + P_{s-1} \tag{3}$$

for all x. Of course the coefficients (one or more) in P_{r-s} and P_{s-1} may vanish (other than the leading one in P_{r-s}). In any case these coefficients $(r - s + 1 + s - 1 + 1)$ or $(r + 1)$ in number can be found by the method of undetermined coefficients by $(r + 1)$ substitutions at most. Usually only those of P_{s-1} are required and these are found by substituting the zeros of P_s in (3) or, in the case of repeated roots, in (3) and in the equalities found from (3) by repeated differentiation with respect to x.

Kingston Grammar School H. G. WOYDA

51. The remainder theorem.

Dr. E. A. Maxwell's Note 2831 (An examination question) calls to mind a proof of the Remainder Theorem which Professor T. M. MacRobert used to show his students in Glasgow.

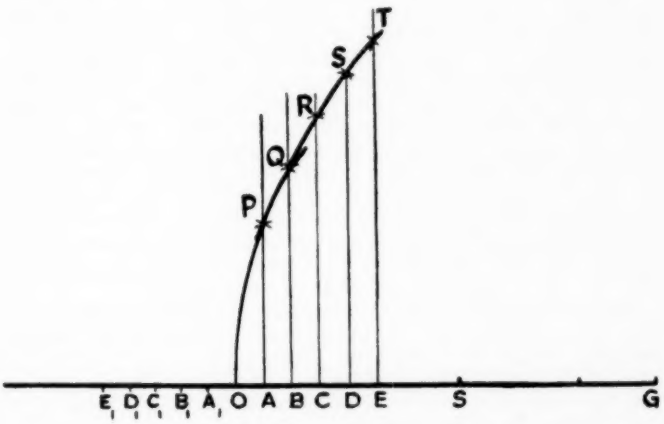
Divide $f(x)$ by $x - a$.

$$x - a) f(x) \qquad (f$$
$$\underline{f(x) - f(a)}$$
$$f(a).$$

Jordanhill College of Education, Glasgow A. G. SILLITTO

52. To describe a parabola given the focus and vertex

Sometimes it pays to let our thoughts stray a little from the direct path. The discussion of a very routine example from a text book led to the following construction:



Let O be the vertex and S the focus. Mark pairs of points A and A_1, B and B_1 etc. at equal distances from O and on OS and SO produced. Erect at A, B, C etc. perpendiculars to OS. Let an arc centre S and radius SA_1, cut the perpendicular at A in P.

Then P is on the parabola. (There are two points.)

Let $OS = a$ and $OA = x$ then $SP = a + x$ and so therefore

$$y^2 = AP^2 = (a + x)^2 - (a - x)^2 = 4ax.$$

N.B. If $AG = 2a$ then PG is the normal at P.

W. H. BEVERSTOCK

REVIEWS

Teaching Mathematics in Secondary Schools. *Ministry of Education Pamphlet.* 36. Pp. 155. 6/-. 1958. (H.M.S.O.)

The difficulties associated with mathematics and its learning have presented a constant challenge to educationists for many years past and never before in the history of school mathematics has there existed so widespread a desire for enlightenment in respect of the subject and of the methods of its teaching. First class reports[1] on the teaching of mathematics at Primary, Secondary and Grammar School levels, prepared by representative groups of teachers, have recently made their appearance, at a time when the voice of the Ministry of Education had not been heard in the field for many years. "Teaching Mathematics in Secondary Schools" fills a gap and comes as a fitting climax to the present round of activity. That the first large issue was sold within a few weeks of publication speaks for itself.

It may be that the raison d'être of the pamphlet is to be found in the opening words of the Minister's foreword. "Our standard of living and our position in the world depend on our ability to remain in the forefront of scientific advance, both pure and applied. Mathematics is the basis of Science..." We have to accept that the chief claim of a subject for educational attention invariably rests upon considerations of utility but that an outlook reinforced by utilitarian interests can nevertheless result in bold adventurousness and strength of purpose. Neither of these is lacking in the pamphlet which is forthright, unambiguously outspoken and free from "padding". Contrary to the impression given by an occasional early press report, the pamphlet is warm, human and encouraging. The views it expresses are clearly the result of mature thought and conviction, based upon the collective experience of several of H.M. Inspectors. The result is a sane, balanced and well-documented piece of work which will do its cause a power of good.

Methods of teaching mathematics to the slower learner have received their full share of attention in recent years but expert tuition is equally necessary for the more able pupil. The pamphlet gives him pride of place in its treatment and asks for the removal of his fetters. We are told, for example, that in the Secondary School, the right answer for new entrants "is to forge ahead with new topics and new ideas" and that "the better girls proceed too slowly". A fifth year spent in revision for the examination is stated to be, for the best pupils, boys and girls, "unproductive of ideas, a period of mathematical stagnation". And at sixth form level, "we become good mathematicians by moving on quickly." "The best and most lasting preparation for a mathematical career at University... is likely to include as much as possible beyond and wide of examination requirements". Again for the Primary School, "the mathematics the child learns should not be confined to arithmetic. The simpler principles of Geometry, Algebra and Mechanics provide

[1] Mathematical Association reports and the I.A.A.M. report, "The Teaching of Mathematics."

suitable material". Insistence on a variety of mathematical experience is thus a recurrent theme in the pamphlet, which is challenging in its implicit demands on the skill and knowledge of the teacher. Reviewers have been quick to notice this. "The chapter on Sixth Form work will force the master or mistress in charge of these studies to the conclusion that he or she is inadequately equipped for the work. To the qualities of the serpent, Job and Solomon which every teacher much bring to his task must be added the versatility of Da Vinci, the application of Newton and the physical stamina of our Olympic champions", is a statement which appears in the Journal of the Association of Assistant Mistresses.[1] The Journal of the National Union of Teachers, "The Schoolmaster and Woman Teacher's Chronicle,"[2] contains an article "X is still unknown" in which it is complained that the pamphlet itself will be largely incomprehensible to the majority of mathematics teachers in Secondary Schools, and that parts of the Primary School section will not be understood by one in a hundred of the teachers. These assertions, if correct, give the measure of the problem the pamphlet seeks to resolve. The author of the article speaks of the mathematical mystique, describing mathematics as a "coterie subject which none but the elect can understand" and he reaches the curious conclusion that "mathematics is too important a subject to be left to the mathematicians.... Their language ought to be made available to the ordinary intelligent people, and there is no reason why it should not be". These words, reminiscent of Pope's "Let mystery to mathematics fly", reveal an unfortunate attitude to mathematics and mathematicians which is more widespread than it ought to be, and which hampers progress. "It is clear that mathematical education has not been entirely satisfactory. The number of adults who feel that there is something in mathematics which has escaped them is shown by the immense sales of books which attempt to justify and explain the subject in simple terms and by the popularity of the simple mathematical problems which appear in newspapers. It must be admitted that Mathematics is not easy to read; it employs a special vocabulary and a symbolic language; like a new game it must be played even at a low level before it can be appreciated and it cannot be understood from the rules alone". The existence of a problem in the teaching of mathematics is presumably largely explained by these words taken from the pamphlet whose publication clearly needs to be followed by strong action on all fronts. The many signs of increased interest in the teaching of mathematics, particularly at the more elementary levels, are encouraging, and the new three year course in the Training Colleges should enable future entrants to the profession to be better equipped mathematically than hitherto.

The pamphlet begins and ends with historical chapters so that the main body of the work is contained in history, so to speak. This gives appropriate expression to the view of the authors that a knowledge of past events in the teaching of mathematics is indispensable to a proper understanding of the present position, and that the subject can be

[1] Vol. X, No. 1. Spring. 1959.
[2] Vol. CLXXV, 2582, January 23, 1959.

adequately taught only against a background of its history. In consequence of this approach, the reader is put in the position of being able to effect a synthesis of the new with the old in teaching, making for soundness of outlook; for to be rooted and grounded in tradition is a necessary and, perhaps, the most effective safeguard against heresy. The chapter, "The History of Mathematics and its Bearing on Teaching", arrives at some significant teaching conclusions. Wave-motion and the triode valve amplifier are suggested as possible topics for mathematical treatment, for example, as are the beginnings of Relativity and some of the newer branches of mathematics, e.g., symbolic logic and Boolean algebra, all this to keep abreast of the times and to help close the gap between school and University.

The teaching of mathematics at the secondary stage and at sixth form level, which forms the core of the pamphlet, is discussed in separate chapters. Specially helpful are the sections on geometry. Stages A, B and C are carefully explained with teaching details, and the present position of the subject, including analytical geometry, is described and clarified. Six groups of sixth form pupils, each requiring separate mathematical treatment, are distinguished, and teaching notes are provided on the various branches of mathematics taken at that level. External examinations and the place of rigour are amongst topics considered, and present day trends are summarised to encourage the use of up-to-date syllabuses. It is pleasing to note the suggestion that the pupil's attention should be drawn to the simplifications (unnoticed in most textbooks) which mathematicians make in applying their subject to the physical world, when assuming the existence of a centre of gravity, for example. The path of a projectile and the laws of friction are also appropriate topics for discussion in this respect. The pamphlet's continual insistence on the need for a thoughtful, enlightened treatment of the subject, and of the pupil, is again evident in a later chapter where it is stated that the mathematical specialist should have some training in learning on his own from books. "It is no kindness to him if his teacher tries to do all his learning for him and continually feeds him with pre-digested material".

In a chapter entitled "In the Classroom", the classroom itself, the library, and such routine matters as homework, tests and marking are discussed. The chapter contains an abundance of practical, common sense suggestions and its keynote is to be found in two sentences—"What is needed is the right atmosphere" and "The teacher of mathematics, for his own profit and enjoyment, not less than for the welfare of his pupils, wants hard work, keen thinking, fluent discussion and, not least, genuine pleasure to come into the lessons".

Other chapters, also offering valuable suggestions for guidance, deal with the relation of mathematics to other school studies, with the underlying principles and methods of teaching mathematics in the Primary School, and with mathematics for the ordinary pupil. The last of these chapters accepts that the ordinary pupil is to be found in all types of secondary schools and a brief section is included which contains suggestions for the teaching of mathematics to the non-mathematical Sixth Former who may enter a Training College. It is

probable that this and the Primary School chapter are intended to be read in conjunction with the relevant Association reports.

One reaches the end of the pamphlet with regret and it would seem ungracious to ask for more. Nevertheless, an interpretation, in terms of educational theory, of the changes in the teaching of mathematics described in the first chapter would have been helpful. Perhaps recent work on concept formation and its relevance to teaching could have been mentioned; also, since there is frequent reference to language aspects of mathematics, one would have liked some analysis of the meaning of the phrase "Mathematics is a language" and of its possible teaching implications. These are important matters which may take us to the heart of mathematics teaching and, though as yet they appear to belong more properly to the field of experiment and research, they are of special interest to all concerned with the professional training of the mathematics teacher. No adequate report at present exists to help those engaged in this work.

The frequent reference to the work of the Association (Mr. Hope-Jones receives special mention) will be a source of satisfaction to members, and the touches of humour in the pamphlet will be appreciated. We are told, for instance, of the C-Form which, under certain circumstances, "may at best take its mathematics examination as a sporting event." We are bidden elsewhere to consider two geometrically similar pigs "as a more properly biological example" illustrating certain mathematical principles. Neither Welsh Whites nor Wessex Saddlebacks, surely, but of *genus mathematicum*! Concisely written with an elegant turn of phrase, the pamphlet has a personal touch often lacking in reports produced by committees. It should be studied by everyone concerned with the teaching of mathematics and will, as a matter of course, become a standard text for students in training.

W. FLEMMING

Mathematics for Higher National Certificate, Vol. 2. By S. W. BELL and H. MATLEY. Pp. x, 486. 35s. 1958. (Cambridge University Press)

This book has been written primarily for students in the second year of a Higher National Certificate Course in Electrical Engineering. It starts with a proof of Taylor's Theorem: this could have been simplified by using the limits 0 and h and condensing the explanatory text: the assumption that $f(x)$ has a continuous nth derivative in the interval might have been stated. The proof of l'Hospital's rule would also benefit by condensation. The chapter ends with the method of expansion in series by use of a differential equation. It is followed by chapters on integration, including the reduction method and miscellaneous properties of integrals, the partial derivative with application to electrical problems and critical values of functions of two variables, complex variable including simple transformations and applications to a.c. networks. In this chapter the authors follow the practice of putting

$z = \cos\theta + j\sin\theta$ in order to expand $\cos^n\theta$ in terms of $\cos n\theta$ etc. This seems quite unnecessary since the relationship

$$\cos\theta = \tfrac{1}{2}(e^{j\theta} + e^{-j\theta})$$

has been previously established. The following chapter, on differential equations, includes a lucid explanation of the method of isoclinals: it also deals with the properties of the D operator, but Particular Integrals are found by assuming the form of the result. The modifications necessary in certain cases are pointed out, but this practice tends, in the case of students, to the adoption of a hit or miss attitude. If time is to be spent on teaching the properties of D, then it is worth while spending further time on the evaluation of

$$\{f(D)\}^{-1} \cdot f(x)$$

In the following chapter there is a very clear introduction to the Laplace transform with application to the solution of ordinary differential equations. The next chapter on Partial Differential equations deals in detail with those in which the variables are separated and with the wave equation in one dimension. Laplace's equation in two dimensions and the diffusion equation are dealt with in the following chapter on the half range form, differentiation and integration of Fourier series. In the chapter on determinants the student is warned that their use in the solution of simultaneous equations is of academic rather than practical interest and an example of solution by the relaxation method is given. This is followed by a chapter on double integrals and one on matrices in which one finds a clear explanation of the rules of matrix algebra, the representation of a complex number in matrix form and the application of matrices to network problems. The notation adopted viz: $(A)_t$ for the transpose and $(A)^{-1}$ for the inverse of (A) has much to recommend it.

The book should prove useful to the students for whom it was written: the authors have taken considerable trouble to anticipate students' difficulties or doubts and, by detailed explanations, to eliminate them. It is very fully indexed, well printed on good paper, contains much information that would normally require reference to books on electrical technology, contains plenty of questions (with answers and sometimes hints) by which a student may test his understanding of the methods described in the text, and is good value.

F. T. CHAFFER

Engineering Mathematics. By R. E. GASKELL. Pp. 462. $7.25. 1958. (Dryden Press. Henry Holt and Co., New York)

Engineers are very fortunate in some of the mathematical texts which have recently been written for them. In quality of production, (paper and printing) ease of reading and breadth of range this is perhaps the best of all of them. Amongst the many topics treated at some length are differential equations, linear algebra, complex function theory, Fourier series, vector algebra and analysis, Special Functions and the Laplace transform.

R. L. GOODSTEIN

Topologie Algebrique et Theorie des Faisceaux. By ROGER GODEMENT. Pp. 283. 3600 Fr. 1959. (Hermann, Paris)

The theory of sheaves (faisceaux) is one of the outstanding developments in mathematics during the last twenty years. A typical example of a sheaf is provided by the germs of differentiable functions on a smooth surface S. Two differentiable functions, defined in neighbourhoods U, V of a point $p \in S$, are called equivalent if they have the same values in some neighbourhood of p, which is contained in the intersection of U and V. The corresponding equivalence classes are called *germs* of differentiable functions at p. Notice that equivalent functions have the same value at p but two functions with the same value at p are not necessarily equivalent. For equivalence it is necessary that they coincide throughout some neighbourhood of p.

In a similar manner one can define the sheaf of germs of vector-fields on S or, more generally of differential forms on a smooth n-dimensional manifold. The theory of sheaves includes what is, perhaps, the best proof of de Rham's theorem that

(a) There is an exact q-form (i.e. one which is locally an exterior derivative) which has given periods when integrated over a set of linearly dependent (piecewise smooth) q-cycles,
(b) a q-form whose integral over every q-cycle has the value zero is an exterior derivative.

Beyond this there are many applications of the theory to topology, algebraic geometry and the theory of complex analytic manifolds.

The theory of sheaves was introduced by J. Leray and a systematic account of it appeared in his great paper on spectral sequences (J. de Math. p. et a., 1950). A more polished account, with several fundamental improvements, was given by H. Cartan in his Paris seminar (1950/51). Further improvements were incorporated in a paper by A. Grothendieck (Tohoku Math. Journal) in 1957.

The book under review is the first to be written on the subject. Algebraic topology and related subjects have been expanding so rapidly during the last fifteen years that any book on an advanced level has been likely to be obsolete before it was printed. The book under review has certainly escaped this fate. In fact, written in the light of "Homological algebra" (Cartan and Eilenberg) and of Grothendieck's paper, it is as timely a book as any I can remember. Moreover it is very good.

In the preface the author indicates that his aim is to make the book accessible to a mathematician who knows nothing of algebraic topology. He has, I think, succeeded in this, though many such readers would probably find it worth while to spend a few days browsing on earlier writings on algebraic topology before tackling this book. I have only one criticism, that there is no index (only an index of notations). This is particularly unfortunate because of the number of ideas which will be new to the kind of reader just envisaged. The book will doubtless run into many editions, and it is to be hoped that this omission will be repaired.

The book consists of Chap. I (pp. 1–106), on homological algebra, Chap. II (pp. 107–267) on sheaves and an appendix (pp. 268–279) on a

further application of the theory of simplicial complexes to the theory of categories and functors and of sheaves.

Chap. I contains, among other things, an excellent account of (complete, semi-) simplicial complexes, also, among the subsidiary topics, the theory of acyclic models. The section on simplicial complexes contains an account of cochains with values in a system of coefficients. This is a natural generalization of the theory of Überdeckung (Reidemeister) or local coefficients (Steenrod). It serves as the basis for the Čech theory of cohomology with coefficients in a stack (presheaf). There is an account of spectral sequences, with special reference to double complexes, and a section on the functors Ext^n, Tor^n.

Chap. II begins with a section on sheaves of sets, that is to say sheaves without algebraic structure, and then proceeds to the theory of sheaves of modules. The cohomology theory of a given sheaf is based on the notion of a "flabby" (flasque) rather than a "fine" sheaf. A sheaf on a space X is flabby if, and only if, every section over each open set in X can be extended to the whole of X. If A is a given sheaf, let $C^0(X, A)$ be the sheaf of germs of possibly discontinuous sections over open sets in X. Then there is a canonical monomorphism $A \to C^0(X, A)$ and $C^0(X, A)$ is flabby. Reiteration of this construction leads to a canonical flabby resolution of A. The cohomology groups of X with respect to A are defined in terms of this resolution. The theory is developed to a satisfying extent and special attention is paid to the Čech theory. There is a section on cup-products. Finally it is proved that every sheaf can be imbedded in an injective sheaf and the resulting theory of derived functors is briefly considered.

J. H. C. WHITEHEAD

Tauberian Theorems. By H. R. PITT. Pp. 174. 37s. 6d. 1958. (Oxford University Press for the Tata Institute of Fundamental Research, Bombay)

Abel's well-known theorem states that the convergence of the series Σa_n to A implies that the sum $A(r) = \Sigma a_n r^n \to A$ as $r \to 1 - 0$. Tauber (1897) proved the converse theorem that, if $A(r) \to A$ as $r \to 1 - 0$ and if $a_n = 0(n^{-1})$, then Σa_n converges to A and, in so doing, gave his name to a whole field of study which has attracted a long list of first-rate mathematicians, notably Hardy, Littlewood, Wiener and the present author. Professor Pitt has written a long series of clear and effective articles on the topic and now presents us with an excellent introduction to the general theory which makes the essence of the subject readily and attractively accessible.

After a short introduction, which makes clear the importance of the generalisation introduced by Wiener, there are treatments of elementary and of classical Tauberian theorems. These are followed by accounts of Wiener's theory and of Mercerian Theorems. The book closes with a chapter on the various proofs of the Prime Number Theorem, in each of which the essential difficulty is tauberian in character and involves deductions about the behaviour of a function from that of its average. Professor Pitt writes with a precise, elegant and easy clarity which adds greatly to the reader's enjoyment.

E. M. WRIGHT

Asymptotic Behaviour and Stability Problems in Ordinary Differential Equations. By L. CESARI. Pp. 271. DM. 68. 1959. (Springer, Berlin)

This is volume 16 of the well-known *Ergebnisse* series. The subject is of interest both theoretically and for many contemporary applications. Written by an acknowledged master in the field, this condensed account of the main results in the very massive literature is extremely welcome.

The bibliography occupies 69 closely printed pages. The title of each paper is followed by a number in italics referring the reader to a particular section of the text, but the text itself sometimes does not refer to relevant papers listed in the bibliography. Thus, one can find readily what sections of the subject a particular mathematician has written about (or, what is effectively the same thing, what sections the author knows that he has written about), but in any section of the text one can find references to some only of the relevant papers listed in the bibliography. This is an unusual arrangement, whose exact purpose is difficult to understand.

The titles of the chapters and sections are:— I. The concept of stability and systems with constant coefficients: §1. Some remarks on the concept of stability, §2. Linear systems with constant coefficients. II. General linear systems: §3. Linear systems with variable coefficients, §4. Linear systems with periodic coefficients, §5. The second order linear differential equation and generalizations. III. Nonlinear systems: §6. Some basic theorems on nonlinear systems and the first method of Lyapunov, §7. The second method of Lyapunov, §8. Analytical methods, §9. Analytic-topological methods. IV. Asymptotic developments: §10. Asymptotic developments in general.

The account of each topic is clear and admirably selective and the depth and extent of scholarship displayed are alike impressive. The excellence of the format and printing are typical of the house of Springer; so also, perhaps, is the somewhat high price. But the wealth of well-arranged information is well worth the money to those interested in the subject.

E. M. WRIGHT.

Un Demi-siècle (1907–1956) de Notes Communiquées Aux Académies. II. Le Champ Réel. By A. DENJOY. Pp. xv, 365, *29*. 1957. (Gauthier-Villars, Paris)

This volume reproduces over 100 papers published by Denjoy between 1907 and 1956. They have been grouped under 16 main headings (B.1 to E.2) and their author has provided a further *29* pages of commentaries relating to them. The pages of the main text are numbered from 225 to 589 following on the numbering in Vol. I. The commentaries for the same reason are on pages numbered *61* to *89*, and are fully provided with cross-references to the main text.

The work is almost wholly Analysis, but in a sense in which Analysis takes analytic topology, ergodic theory, and the calculus of probability in its stride. The list of topics is impressively wide, but the really

phenomenal character of the volume lies in the depth of almost every page. I do not know whether Professor Denjoy ever shrank from a difficult problem; but this volume shows that many difficult problems shrank before Professor Denjoy. They did not shrink up to nothing; no-one who insists on understanding what he reads will find this book easy. But any mathematical analyst will find in it a whetstone for his intellect which will last a long time.

It is over 50 years since Denjoy published his first paper. His last has not yet been written. This remarkable mathematician came to Edinburgh in 1958 to give an account of some of his recent work to the International Congress.

H. D. URSELL

Ancillary Mathematics. By H. S. W. MASSEY and H. KESTELMAN. Pp. xvi, 990. 75s. 1959. (Pitman and Sons Ltd.)

This book is more in the spirit of the great French *Cours D'Analyse* than its title would lead one to expect. All too often a very large book on mathematical techniques is a collection of indifferent to bad pieces of mathematics because no one writer excels over the whole range of topics it discusses, but this book is an outstanding exception; the mathematics discussed is ancillary only in the sense that it is mathematics which the scientist cannot dispense with, not because it is itself in some way inferior. Far too seldom has a book on mathematics for scientists combined rigour and simplicity as well as has been achieved in this volume. It would need a whole review section of the Gazette and not just an essay to do justice to the manner in which the authors have achieved their success. Amongst the many topics studied are the theory of functions of one and more variables, differential geometry in two and three dimensions, motion of a particle and rigid dynamics (including Lagrange's equations) Fourier Series, and the Calculus of Variations. To the worked examples have been added an immense collection of problems, the solution of which has been a co-operative effort of the mathematics department at University College, London.

Most of the results given in the book have been proved in detail, and many of the proofs are new or interesting variations of standard results. Where the best result in the field is not susceptible of proof by the methods available in the book the authors have not hesitated to give the result without proof, contenting themselves with a very careful statement of the conditions under which it holds. For example, the dominated convergence theorem (p. 434) that

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim f_n(x)\,dx$$

if $f_r(x)$, $r = 1, 2, 3, \ldots$ are defined for $a < x < b$ and satisfy $|f_r(x)| < \phi(x)$, and if $\int_a^b \phi(x)\,dx$ exists.

In the case of the theorem on substitution in a definite integral, however, (p. 402), given in the form

$$\int_{G(a)}^{G(b)} f(t)\,dt = \int_a^b f\{G(x)\}g(x)\,dx, \quad G(x) = \int_a^x g(t)\,dt,$$

the conditions have been overlooked and it would appear to be necessary either that $g(t)$ be of constant sign in (a, b) or that $f(t)$ be continuous between the lower and upper bounds of $G(x)$ in $a < x < b$.

There is a very interesting formula for multiple integrals

$$\int_R F\{f(x, y, z)\}\,dx\,dy\,dz = \int_m^M F(t)\,\frac{dV}{dt}\,dt$$

where $V(t)$ is the volume of that part of R for which $f(x, y, z) < t$ and m, M are the extreme values of f in R, which is skilfully exploited in the evaluation of such an integral as

$$\int_R f(r_1, r_2, r_{12})\,dx_1\,dy_1\,dz_1\,dx_2\,dy_2\,dz_2$$

where r_1, r_2, r_{12} are the distances OP_1, OP_2, P_1P_2 respectively and P_1, P_2 are the points with rectangular coordinates (x_1, y_1, z_1), (x_2, y_2, z_2) respectively.

In a book intended for Physicists it was a little surprising to find how little space had been given to differential equations and it is perhaps worth noting that better methods are available for handling simultaneous systems than those which are given. The entire omission of matrices is also rather surprising, but of course something has to be left out. It would certainly seem worth while, in the next edition, to add just a few pages to the chapter on complex functions to bring in the theory of residues, for this would effect a considerable saving of space and effort in the evaluation of many famous definite integrals.

R. L. GOODSTEIN

The Structure of Arithmetic and Algebra. By M. H. MARIA. Pp. 294. 48s. 1958. (Chapman and Hall)

The author's object is to build up the theory of real numbers on an axiomatic foundation without using the terminology of modern algebra or the symbolism of mathematical logic, to make the text available to a very wide class of students. The aim is an excellent one and has been very successfully accomplished. The book is very easy to read, sufficiently reliable for its purpose and containing a good many elementary results not readily available elsewhere, for instance a very clear proof of the validity of the familiar square root process of elementary arithmetic which is generally—and mistakenly—supposed to rest simply upon the corresponding algebraic process.

One primary difficulty in the axiomatic construction of the number system is to reconcile the natural numbers defined in the formalism with the natural numbers used in the descriptive text (and even those at the head of the page). Dr. Maria does not draw the reader's attention to this problem, and I think that she should. To integrate the natural numbers defined axiomatically with the numbers of everyday usage

what is needed is the incorporation of a theory of finite classes into the axiomatic development, or what comes to the same thing, the incorporation of a counting operator into the system (as for instance in the introduction to the reviewer's *Recursive Number Theory*).

R. L. GOODSTEIN

Tables of Partitions. Initiated and computed by HANSRAJ GUPTA, computed independently by C. E. GWYTHER and J. C. P. MILLER. Pp. xxxix, 132. 63s. 1958. Royal Society Mathematical Tables, Volume 4. (Published for the Royal Society at the University Press, Cambridge.)

The main table in this beautifully printed volume is Table I (pages 2–88), which gives values of $p(n, m)$, the number of partitions of n into at most m parts (or, alternatively, into parts of which the greatest is at most m). Several independent computations were made by means of the recurrence relation

$$p(n, m) = p(n, m - 1) + p(n - m, m)$$

used by Euler to construct a small table, and other formulae were used for the purpose of checking. The values of $p(n, m)$ are given for $n = 1(1)100$, $m = 1(1)n$; $n = 101(1)200$, $m = 1(1)100$; $n = 201(1)400$, $m = 1(1)50$. The table thus includes the values of $p(n)$, $= p(n, n) =$ the number of unrestricted partitions of n, up to $n = 100$, and these values are printed in bold type. The values for fixed n are arranged in columns. When $m \geqslant n$, $p(n, m)$ evidently equals $p(n, n)$, and up to $n = 100$ the columns of values do in fact extend beyond $m = n$; they are completed by repetitions of the values of $p(n, n)$, in order that there may be no blanks to be misconstrued as zeros.

There are three auxiliary tables involving quantities $p_s(n, m)$ which may be defined by

$$\prod_{r=1}^{m} (1 - t^r)^{-s} \prod_{r=m+1}^{\infty} (1 - t^r)^{-s+1} = \sum_{n=0}^{\infty} p_s(n, m) t^n.$$

The most extensive is Table II (pages 90–121), which gives values of $p_2(n, m)$ for $n = 0(1)50$, $m = 0(1)n$; $n = 50(1)550$, $m = 0(1)M$, where M gradually decreases from 23 to 1; $n = 550(1)1000$, $m = 0$. The number of unrestricted partitions of n is here included, as $p(n) = p_2(n, 0)$, up to $n = 1000$, compared with an upper limit of $n = 100$ for $p(n)$ in Table I. Table III (pages 122–131) gives values of $p_3(n, m)$ up to $n = 200$ for various m, and Table IV (page 132) gives values of $p_4(n, 0)$ up to $n = 200$. By formulae which are given, each table of $p_s(n, m)$ may be used to calculate, verify or extend the preceding table of $p_{s-1}(n, m)$.

Among checks on the auxiliary tables were programmes constructed by A. J. Keeping for *EDSAC*, the electronic computer at Cambridge. The programme relating to Table II used quadruple-length arithmetic, as the largest number involved, namely $p(1000)$, contains 32 decimal digits.

The tables are preceded by a lengthy mathematical introduction which sets up formulae and methods of calculation and works out numerical examples. There are also a description of the tables and a bibliography of some 80 items.

It is clear that great pains have been taken to ensure the accuracy of the tables and that the whole volume is the result of considerable labour. It is likely to remain standard for a long time.

A. FLETCHER

Applied Analysis. By CORNELIUS LANCZOS. Pp. 539. 55s. 1957. (Pitman)

The author sets out to show 'the secret life behind the Golem-like appearance' of mathematical formulae. He has succeeded admirably and every teacher of numerical analysis should take note of his methods. Now that it is fashionable to emphasize the abstract aspect of things it is very refreshing to meet a book in which an attempt is made to explain the ideas and point of view of the practical mathematician. The author's long experience as a user as well as an inventor of numerical methods enables him to provide much important advice for the would-be solver of practical problems, the methods he describes being often those to whose development he has made considerable contributions.

The book is clearly printed, the formulae and examples (which are plentiful) being well displayed. Unfortunately, the proof-readers have permitted numerous small misprints and occasional incomprehensible sentences to pass. It may be hoped that these will be corrected in the future editions. There are also several places in which the difference between American and English usage may cause trouble to students; in the introduction there is, perhaps, an instance of this. Here. the author interprets 'Numerical Analysis' to mean the study of rounding errors, and invents a new name—Parexic Analysis—for that part of the theory of approximation that would be claimed as their proper field of work by most numerical analysts in this country.

The titles of the chapters give no hint of the wealth of understanding that is to be found in the book, and the detailed table of contents occupies six pages, so a brief indication of some of the more important topics that are clarified by the author is not out of place.

The first chapter is rather weaker than the others. It gives an account of some methods of solving algebraic equations, particular emphasis being given to Newton's method, and Bernoulli's method, use being made of harmonic analysis to determine the phases of complex roots. The statement, on page 17, that Horner's technique of repeated division is not of use for the evaluation of $f'(x)$ when a quadratic divisor is used in the search for a complex root, is wrong.

Chapter Two occupies about one hundred and twenty pages, and contains an extraordinarily illuminating account of matrix ideas and methods. One of the principal tools the author uses is the idea of a skew frame of reference together with its adjoint frame, and with this he makes clear the geometric significance of many of the properties of unsymmetric matrices. Chapter Three is entitled Large-Scale Linear

Systems and deals with some of the practical difficulties that arise when large sets of equations are considered It is now known that some further difficulties arise in practice, and Wilkinson's two recent papers[1] could well be consulted before numerical work is undertaken.

The next chapter, entitled Harmonic Analysis, deals with trigonometric interpolation, and describes a method of increasing the convergence of the partial sums which is valuable when it is necessary to differentiate a series. The numerical inversion of the Laplace transform is considered in some detail, and an explanation of the unsatisfactory nature of processes for fitting exponentials to tables is given. The search for hidden periodicities is also dealt with fully.

Chapter Five is about Data Analysis. It contains an indication of a few common finite difference formulae, quite a lot about smoothing and smoothed differentiation, a short account of orthogonal functions and polynomials, and a resumé of Runge's work on the convergence of equidistant polynomial interpolation.

The chapter on Quadrature Methods deals mainly with Gaussian and Eulerian type formulae, and some examples of their use to solve differential equations, and to obtain rational approximations to functions are given.

The last chapter deals with methods of obtaining rapidly convergent series representations of functions defined by their Taylor series or by differential equations.

There can be no doubt that this is a valuable contribution to the literature of numerical analysis, not only for students but for research workers, to whom it will be a source of delight and inspiration.

S. MICHAELSON

Multivalent Functions. By W. K. HAYMAN. Cambridge Tracts in Mathematics and Mathematical Physics, No: 48. Pp. 151. 27s. 6d. 1958. (Cambridge University Press)

This Cambridge Tract on one of the most difficult fields of analytical research is remarkably well written. and this cannot have been an easy task for this author. He is, of course, one of the leading experts in this field, and we can expect from him a highly competent account, but we must be grateful to him for the great care he has taken to make his account readable and reliable even to the smallest technical detail of proof. This is not to say that this Tract is an easy book. Its study presupposes mature competence in mathematical analysis, but the serious reader will be rewarded not only by the rather specialised but fascinating content but also by the general skilful techniques of methods (areal principle, symmetrization) he can learn from it. The title "Multivalent Functions" really means "Mean p-valent Functions", that is, functions $f(z)$ regular in $|z| < 1$, say, that take any value w at most p times, in some specified mean sense. They were first introduced by Spencer (1940), but their theory of distortion, order, and associated coefficient problems is a generalization of the earlier classical theory of

[1] J. H. Wilkinson, Computer Journal, 1, 1958. pp. 90–96 and 148–152.

"schlicht" functions as developed by Koebe, Bieberbach, Löwner and their pupils about 25 years before. The tract covers all these theories in their essentials, with the exception of the more recent variational methods of Jenkins and Schiffer.

W. W. ROGOSINSKI

Normed Linear Spaces. By M. M. DAY. Pp. 139. DM 28. 1958. (Springer, Berlin)

This book is well described by a sentence in the Foreword which states that it is "a compressed introduction to the study of normed linear spaces and to that part of the theory of linear topological spaces without which the main discussion could not well proceed".

The discussion is indeed compressed—a vast amount of material is concentrated into 121 pages of text. The style is the rapid fire Definition —Theorem—Proof kind, with no space "wasted" by any asides. This method is appropriate for a Report while a more leisurely, relaxed, treatment is needed for a textbook (e.g., Kolmogorov and Fomin, Functional Analysis, vol. 1, Metric and Normed Spaces). Day's book is an excellent Report on the status of normed linear spaces. A Reader's Guide refers to the literature for those subjects that the author mentions only briefly in the text or not at all. Over 120 contributors are listed in the Bibliography, many of whom have contributed several papers. The reviewer notes that no mention whatever is made of the work of N. Aronszajn, who obtained in 1935 (Comptes Rendus de l'Acad. Sci., Paris, vol. 201, pp. 811–813; pp. 873–875) a norm characterization of inner product spaces among normed linear spaces that the author calls, "the most memorable" of criteria due to E. R. Lorch—but then Lorch's paper of 1948 did not refer to Aronszajn either!

LEONARD M. BLUMENTHAL

Combinatory Logic. By H. B. CURRY and R. FEYS. Pp. 417. 42s. 1958. (North Holland Publishing Co., Amsterdam)

This important book is the first of two volumes surveying the whole field of Combinatory Logic. The senior of the two authors, H. B. Curry, was the founder of the subject together with M. Schönfinkel. Combinatory logic, as the authors remark in their introduction, aspires to be an analysis of the ultimate foundations. It derives from an analysis of the process of substitution for variables.

The first part of the work discusses the general notion of a formal system, variables, substitution, and proof theory. This is followed by accounts of calculi of lambda conversion (Church calculus), and of an intuitive theory of combinators and its formalisation; the book concludes with a study of the theory of functionality. Two short sections, one on the definitional independence of combinators and the other on some aspects of the transition from non-logistic to logistic systems are the work of W. Craig.

The book is noteworthy for the great care and attention to detail with which it is written and for its lucid exposition of difficult ideas.

R. L. GOODSTEIN

Introduction to the Problem of Minimal Models in the Theory of Algebraic Surfaces. By O. ZARISKI. Pp. 90. $2. 1958. (The Mathematical Society of Japan)

Modern algebraic geometry has today reached the stage at which it can confidently set about reestablishing and extending the classical results of the Italian school. The theorems of Enriques and Castelnuovo on the classification of algebraic surfaces present an obvious challenge in this direction, and the challenge has been taken up in this monograph by Professor Zariski.

The book is divided into three parts. The first two occupy all but 7 pages and deal with rational transformations and exceptional curves. The final part then sketches the proof of the main theorem on minimal models (details are available in the American Journal of Mathematics). The book is thus mainly devoted to the development and application of certain basic techniques of modern algebraic geometry. As such it should prove useful to algebraic geometers, but its general appeal will be limited. The style is succinct but not unintelligible.

M. F. ATIYAH

Introduction à L'Étude des Variétés Kählériennes. By ANDRÉ WEIL. Pp. 175. F. 2000. 1958. (Hermann, Paris)

The theory of complex manifolds, and more especially of Kähler manifolds, is a very flourishing branch of current mathematics. The fundamental pioneering work was done by Hodge twenty years ago, and his book on Harmonic Integrals has until now been the only one on the subject. Professor Weil's book is essentially a modernised version of the Hodge Theory. It contains nothing basicly new, but it is more up to date in its point of view.

The first few chapters deal with the following topics: exterior algebras, local Kählerian geometry, harmonic forms. The treatment is elementary and self-contained, but the proof of the existence theorem for harmonic forms is not given. The last part of the book is devoted to complex tori and theta functions. This provides an admirable application of the theory developed in the first half of the book, and gives the author an opportunity of putting the classical theory of Abelian varieties into modern dress.

This is a very elegant book and should provide an excellent introduction to the subject. It stops short of the theory of sheaves, but this was no doubt inevitable in a work of this type. The quality of the printing will come as a very pleasant surprise to those familiar with older French publications.

M. F. ATIYAH

Lie Groups. By P. M. COHN. Cambridge Tracts No: 46. Pp. 160. 22s. 6d. 1957. (Cambridge University Press)

This is a lucidly written introduction to one of the finest and most imposing edifices modern mathematics has erected. It is inherent in the theory of Lie Groups, both historically and systematically, that it

requires a heavy apparatus from several branches of mathematics. The first two chapters of the book give an account of this background material from analysis, topology and algebra. They deal with the elements of the theory of analytic manifolds, infinitesimal transformations and their differential forms; they then introduce the fundamental definition of topological groups, local groups, and Lie groups. In Chapter 3 the operation of commutation of infinitesimal transformations is introduced and the Lie algebra associated with a Lie group is constructed as the algebra of infinitesimal right translations of the group. Chapter 4 develops the operations on differential forms and culminates in the derivation of the Maurer–Cartan equations. These are used in Chapter 5 to prove the existence and uniqueness of a local Lie group having a given Lie algebra. (The more recent global aspect of the theory is not included.) The requisite facts from the theory of differential equations are proved in an Appendix. Chapter 6 studies the connections between subgroups (subalgebras) and homomorphisms of Lie groups (algebras). In particular, it is proved that every continuous homomorphism of a Lie group is analytic and that every closed subgroup of a Lie group is itself a Lie group. The final chapter contains the construction of a universal covering group for every connected Lie group.

The reader of this slim book of 160 pages is well equipped to tackle one of the more detailed accounts of our present knowledge of Lie groups such as Chevalley's well-known three volumes which, of course, go considerably further in depth and width.

K. A. HIRSCH

BRIEF MENTION

The Foundations of Euclidean Geometry. By H. G. FORDER. Pp. 349. $1.50. 1958.

The Elements of Non-Euclidean Geometry. By D. M. Y. SOMMERVILLE. Pp. 274. $1.50. 1958.

An Introduction to the Geometry of N-Dimensions. By D. M. Y. SOMMERVILLE. Pp. 196. $1.50. 1958. (Dover, New York).

These three fine additions to the Dover Series need no recommendation. Forder's insight lights up the pages of his important study of the foundations of geometry. Sommerville's books provide what are probably the student's easiest introduction to non-Euclidean and n-dimensional geometry.

A Dictionary of Named Effects and Laws in Chemistry, Physics and Mathematics. By D. W. G. BALLENTYNE and L. E. Q. WALKER. Pp. 205. 30s. 1958. (Chapman and Hall)

Some of the mathematical entries are very illuminating and complete in a few lines, others, like the entry under "Routh's Rule", are of little value. Mathematical *definitions* sampled at random were not all entirely correct (see for instance *Dedekind*; *Raabe's test*).

Guide to the Literature of Mathematics and Physics, Including Related Works of Engineering Science. By N. G. PARKE. Pp. 436. $2.75. 1958. (Dover, New York)

The guide discusses the literature under specific headings like 'Geometry', 'Functional Analysis', 'Calculus' and closes each section with a book list. There are both author and subject indexes to add to the value of this very useful work.

Linear Programming and Associated Techniques. By VERA RILEY and S. I. GASS. Pp. 613. 48s. 1958. (Johns Hopkins University Press and Oxford University Press)

This is not only a comprehensive guide to the literature but includes critical appraisements of the papers and books listed.

Proceedings of the VIII-th International Astronautical Congress. Pp. 607. 170s. 1958. (Springer, Vienna)

This magnificent production contains more than 45 lectures in English, German, Italian, French and Russian (with English summaries) the majority being in English. The Editorial Board includes such well-known scientists as W. von Braun. The lectures range over many topics, theory of flight, relativistic rocket mechanics, biology, space law and speculation on the evolution of life on other planets. The Russians report information on cosmic radiation gained by using earth satellites. The only contribution from this country is by W. N. Neat, on the problem of variable thrust. The Congress was held in Barcelona in October, 1957, starting just two days after the launching of Sputnik I on October the 4th.

THE MATHEMATICAL ASSOCIATION

The fundamental aim of the Mathematical Association is to promote good methods of Mathematical teaching. Intending members of the Association are requested to communicate with one of the Secretaries. The subscription to the Association is 21s. per annum and is due on January 1st. Each member receives a copy of the *Mathematical Gazette* and a copy of each new Report as it is issued.

Change of address should be notified to the Membership Secretary, Mr. M. A. PORTER. If copies of the *Gazette* fail to reach a member for lack of such notification, duplicate copies can be supplied only at the published price. If change of address is the result of a change of appointment, the Membership Secretary will be glad to be informed.

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The address of the Association and of the Hon. Treasurer and Secretaries is **Gordon House, 29 Gordon Square, London, W.C.1.**

BRANCH REPORTS

SOUTHAMPTON AND DISTRICT BRANCH

Annual Report for 1958–1959

The branch has been pleased to see a slight increase in membership this year although the average attendance remains in the middle forties.

Friday, 24th October, 1958. Mr. A. P. Rollett gave a talk entitled "The Shortage of Mathematicians". The former "rigour first" approach in schools should be more and more replaced by the greater interest of the heuristic approach in spite of the difficulties caused by the vicious triangle of syllabus, examination and text book. Cross-setting would help the mathematically able to mature earlier and would greatly assist the slower worker.

Thursday, 27th November, 1958. The Annual General Meeting of the Branch elected the following to the Committee:— President: Mrs. W. S. P. Edmunds, Vice-President: Professor E. T. Davies, University Representative: Miss N. Walls, Secretary and Treasurer: Mr. J. C. F. Fair, and also Miss H. Bromby, Miss R. A. Clarke, and Mr. T. A. Jones. Mrs. Edmunds then gave her Presidential Address entitled "Gleanings of Mathematical teaching in the United States of America". This was necessarily a comparison with mathematical teaching "over here" where we advance on a broad front gradually learning more about each mathematical subject rather than "their" individual sorties to mop up each separate subject one at a time.

Thursday, 29th January, 1959. Mr. S. Weintroub gave an illustrated lecture on "Some achievements of the International Geophysical Year", in which he indicated the vast field covered in this gigantic co-operative effort involving 67 nations. He gave some early results of some of the observations and stated that enough material had been gathered to keep us busy for a century.

Tuesday, 3rd March, 1959. Professor T. A. Broadbent spoke on "The Mathematical Association—What it is and does". This excellent meeting was mainly for the purpose of reminding members and informing non-members of the Association's objects so that the Annual Conference, a month later, would be staffed by sufficient intelligent hosts. Professor Broadbent's talk encouraged the Branch greatly.

Friday, 29th May, 1959. Dr. T. M. Flett spoke "On Tangents to Curves". He criticised certain well-known definitions and theorems showing us that even common assumptions must not for ever pass unexamined.

Thursday, 18th June, 1959. Dr. R. S. Clark gave a lecture about "Knots". The "Knot Problem" was stated and the invariant polynomial of a knot discussed in detail. Both the content and the presentation of the lecture were fascinating.

J. C. F. FAIR
(Hon. Sec.)

THE LIVERPOOL MATHEMATICAL SOCIETY

REPORT FOR THE SESSION 1958–59

20th October, 1958. The Society's Prize for 1958 was presented to Mr. J. P. Wilson. Mr. G. F. Peaker of the Ministry of Education addressed the Society on "Mathematics as part of a liberal education".

17th November, 1958. Dr. Mary Cartwright addressed the Society on "Mathematics in Russia".

8th December, 1958. Mr. R. L. Plackett gave a lecture entitled "Life Testing".

12th January 1959. Dr. Martyn Cundy of Sherborne School gave a lecture entitled "The Mathematics of AND and OR" with demonstrations of electrical devices for performing simple logical and arithmetical calculations.

21st February, 1959. Members of the Society visited the Liverpool Observatory and Tidal Institute.

9th March, 1959. Members' Night. The following papers were read by members of the Society.

Mrs. M. J. E. Farnell: Ground Resonance Investigations on a Helicopter.

Mr. D. Hill: The Gravitational Field of a Rotating Body.

Mr. W. M. Brookes: The Problem of Teaching Mathematics.

11th May, 1959. The President chose for his address to the Annual General Meeting the title "The Teaching of Mathematics". The following members were elected to serve on the Council during the Session 1959–60.

President, Dr. P. C. Davey; Vice-President, Dr. G. R. Baldock; Treasurer, Dr. W. F. Newns; Secretary, Dr. M. C. R. Butler.
Committee: Mr. W. M. Brookes, Dr. T. M. Flett, Mr. J. Kershaw, Miss A. H. Ainsworth, Mr. J. P. Coles, Miss V. M. Hughes, Dr. T. J. Willmore.

There are about 100 members of the Society, and the average attendance at meetings is about 60.

N.S.W. BRANCH

Report on Activities for 1959

During 1959 seven meetings of the N.S.W. Branch of the Mathematical Association were held. With secondary education in N.S.W. facing a major reorganization as a result of the Wyndham Report, guest speakers at meetings have treated various aspects of the problems of a modern mathematical education.

Addresses were given as follows:

12th June. Professor Neumann (of Manchester University), on "The General School Certificate Examination in England".

31st July. Lady Jeffreys, on "The Conflict of Studies".

18th September. Professor Bosson (from the University of N.S.W.), on "Mathematics as a Social Science".

16th October. Professor Stone (from the University of Chicago), on "The World Looks at the Problems of Mathematical Education".

In addition, two meetings have been devoted to consideration of the organization of Mathematics courses in N.S.W. secondary schools.

At 30th September 1959 there were 921 names listed as Subscribers to the Journal of the Branch, "The Australian Mathematics Teacher".

At the annual general meeting of 30th October, the following office-bearers for 1960 were elected:

President: Dr. I. S. Turner.
Vice-Presidents: Miss I. Barnes, Mr. J. D. Graham.
Hon. Secretaries: Miss D. Brownell, Mr. A. R. Bunker.
Hon. Treasurer: Mr. A. Aitkin.
Assistant Treasurer: Mrs. N. Hughes.
Director, Problems Bureau: Mr. R. J. Gillings.
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Chairman, Editorial Committee: Mr. J. L. Williams.

A. R. Bunker
(Hon. Secretary)

YORKSHIRE BRANCH

Report for the Session 1958–59

The following meetings were held:—

11th October, 1958. Annual General Meeting. Mr. W. Taylor, Retiring President, spoke on "Preparation for the Teaching of Mathematics".

15th November, 1958. Mr. J. S. Barnshaw and Mr. H. H. Watts introduced a discussion on "The Mathematical Association's Report on the Teaching of Algebra in Sixth Forms".

28th February, 1959. Dr. S. Tempest spoke on "Space Travel".

9th May, 1959. Prof. G. S. Rushbrooke of King's College, Newcastle, spoke on "Experimental Mathematics".

The average attendance at these meetings was 46.

During the Spring Term, 1959, four evening lectures on "The Special Theory of Relativity and its Applications", were given by Dr. N. B. Slater of Leeds University and were very well attended.

At the Annual General Meeting, 1958, Prof. T. G. Cowling, F.R.S., was elected President, 1958–60.

L. K. TURNER
(Hon. Secretary)

T. L. HEATH

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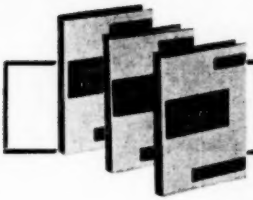

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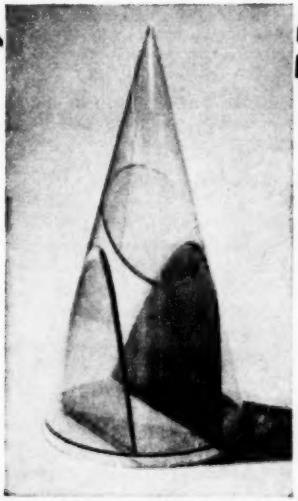

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